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# Structural properties of the graph algebra $K_3$

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**Abstract**

The algebras $Q_n$ describe the relationship between the roots and coefficients of a non-commutative polynomial. I. Gelfand, S. Gelfand, and V. Retakh have defined quotients of these algebras corresponding to graphs. In this paper we find the Hilbert series of the class of algebras corresponding to the graph $K_3$ and show that this algebra is Koszul.


*MSC:* 16S37

## 1. Introduction

Let $P(x) = x^n - a_{n-1}x^{n-1} + a_{n-2}x^{n-2} - \cdots + (-1)^n a_0$ be a polynomial over a division algebra. Gelfand and Retakh [3] studied relationships between the coefficients $a_i$ and a generic set $\{x_1, \ldots, x_n\}$ of solutions of $P(x) = 0$. For any ordering $(i_1, \ldots, i_n)$ of $\{1, \ldots, n\}$ one can construct *pseudoroots* $y_k$, $k = 1, \ldots, n$, (certain rational functions in $x_{i_1}, \ldots, x_{i_n}$) that give a decomposition $P(t) = (t - y_n) \cdots (t - y_2)(t - y_1)$ where $t$ is a central variable.

In [4] Gelfand, Retakh, and Wilson introduced the algebra $Q_n$ of all pseudoroots of a generic non-commutative polynomial, determined a basis for this algebra and studied its structure. The algebras $Q_n$ have a presentation given by generators $u(A)$, $\emptyset \neq A \subset [n]$ and relations

$$\sum_{C,D \subset A} [u(C \cup i), u(D \cup j)] = \left( \sum_{E \subset A} u(E \cup i \cup j) \right) \sum_{F \subset A} (u(F \cup i) - u(F \cup j))$$

for all $A \subset [n]$, $i, j \in [n] \setminus A$, $i \neq j$.

In [2] Gelfand, Gelfand, and Retakh introduced a class of quotient algebras of $Q_n$ corresponding to graphs on $n$ nodes. Let $G$ be a graph with vertex set $[n] = \{1, 2, \ldots, n\}$ and edge set $E$ composed of elements of $P([n])$ with cardinality two (hence $G$ has no loops of multiple edges). We can then consider the quotient algebra $Q_n(G)$ that we get by adding the additional relations $u(\{i, j\}) = 0$ if $\{i, j\} \notin E$ to $Q_n$. The following theorem gives a nice presentation of the algebra $Q_n(G)$.

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**Theorem 1.1** (*[2]*). *Let $G$ be a graph on $n$ nodes with edge set $E$. Then the algebra $Q_n(G)$ is generated by the elements $u(i)$ for $i \in [n]$ and $u(i, j)$ for $\{i, j\} \in E$ with the following relations (assume $u(i, j) = 0$ if $\{i, j\} \notin E$):*

(i) $[u(i), u(j)] = u(i, j)(u(i) - u(j))$ $i \neq j, i, j \in [n]$
(ii) $[u(i, k), u(j, k)] + [u(i, k), u(j)] + [u(i), u(j, k)] = u(i, j)(u(i, k) - u(j, k))$ *for distinct* $i, j, k \in [n]$
(iii) $[u(i, j), u(k, l)] = 0$ *for distinct* $i, j, k, l \in [n]$.

We wish to consider the algebra that is generated by the graph $K_3$. By Theorem 1.1 this algebra has generators $u(1), u(2), u(3), u(12), u(13), u(23)$ together with the following relations which we will refer to as $r_1$ through $r_5$, in $V \otimes V$ (where $V$ denotes the span of the generators).

$$
\begin{aligned}
r_1 &= [u(1), u(2)] + u(12)(u(2) - u(1)) = 0\\
r_2 &= [u(2), u(3)] + u(23)(u(3) - u(2)) = 0\\
r_3 &= [u(3), u(1)] + u(13)(u(1) - u(3)) = 0\\
r_4 &= [u(12), u(23)] + [u(12), u(3)] + [u(1), u(23)] - u(13)(u(12) - u(23)) = 0\\
r_5 &= [u(12), u(13)] + [u(12), u(3)] + [u(2), u(13)] - u(23)(u(12) - u(13)) = 0.
\end{aligned}
$$

We have only five relations because all other possible combinations in (i) and (ii) are linear combinations of these five. We also have no relations of type (iii) because $n = 3$ and we do not have four distinct integers to work with.

Our study of $K_3$ requires us to examine its associated graded algebra under a particular filtration. We construct an increasing filtration on $K_3$ by defining $F_n$ to be the span of all monomials $u(A_1)u(A_2)\cdots u(A_k)$ such that $\sum_{i=1}^{k} | A_i | \leq n$. It is clear that our $F_i$ are subspaces with the properties $\bigcup_i F_i = K_3$ and $F_i F_j \subseteq F_{i+j}$. We set the define $F_0$ to be the span of 1.

Now we form $\mathrm{gr}(K_3)$ in the usual way. Take $G_i = F_i / F_{i-1}$ and set $\mathrm{gr}(K_3) = \bigoplus_i G_i$ and then define multiplication in $\mathrm{gr}(K_3)$ so for all $a \in F_i, b \in F_j$, $(a + F_{i-1})(b + F_{j-1}) = ab + F_{i+j-1}$. Note that there is a non-linear map $gr : K_3 \to gr K_3$ that sends $a \in F_i, a \notin F_{i-1}$ to $a + F_{i-1}$ in $\mathrm{gr}(K_3)$ and sends 0 to 0.

Our main goal is to show the algebra $K_3$ has the Koszul property. To do this we first find a presentation of $\mathrm{gr}(K_3)$. We next show it is enough to prove that the algebra $\mathrm{gr}(K_3)$ is Koszul. Finally we will work with lattices of vector spaces arising from $\mathrm{gr}(K_3)$ to complete the proof that $K_3$ is Koszul.[1]

## 2. Koszul algebras

There are a number of equivalent definitions of Koszul algebras including this lattice definition from Ufnarovskij [10].

**Definition 2.1.** A quadratic algebra $A = \{V, R\}$ (where $V$ is the span of the generators and $R$ the span of the generating relations in $V \otimes V$) is Koszul if the collection of $n - 1$ subspaces $\{V^{\otimes i-1} \otimes R \otimes V^{\otimes n-i-1}\}_i$ generates a distributive lattice in $V^{\otimes n}$ for any $n$.

In [5] the following criterion is given for distributivity of a modular lattice:

**Theorem 2.1.** *Suppose $\{x_1, \ldots, x_n\}$ generates the modular lattice $\Omega$. If any proper subset of $\{x_1, \ldots, x_n\}$ generates a distributive sublattice then $\Omega$ is distributive iff for any $2 \leq k \leq n - 1$ the triple $x_1 \vee \cdots \vee x_{k-1}, x_k, x_{k+1} \wedge \cdots \wedge x_n$ is distributive.*

Applying Theorem 2.1 to Definition 2.1 we get the following corollary which we will use throughout this work:

**Corollary 2.1.** *The quadratic algebra $A = \{V, R\}$ (where $V$ is the span of the generators and $R$ the span of the generating relations in $V \otimes V$) is Koszul if $RV^{n-2} \cap VRV^{n-2} \cap \cdots \cap V^{a-2}RV^{n-a}$, $V^{a-1}RV^{n-a-1}$, $V^a RV^{n-a-2} + \cdots + V^{n-2}R$ is a distributive triple in $V^n$ for any $a$ and $n$ with $2 \leq a \leq n - 2$.*

[1] This work was originally completed in [6]. The author then learned that this also follows from independent work of Piontkovski. His results have appeared in [7].