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A-infinity structure on Ext-algebras

D.-M. Lu [a], J.H. Palmieri [b,*], Q.-S. Wu [c], J.J. Zhang [b]

[a] Department of Mathematics, Zhejiang University, Hangzhou 310027, China
[b] Department of Mathematics, Box 354350, University of Washington, Seattle, WA 98195, USA
[c] Institute of Mathematics, Fudan University, Shanghai, 200433, China

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ABSTRACT

Let A be a connected graded algebra and let E denote its Ext-algebra $\bigoplus_i \mathrm{Ext}^i_A(k_A, k_A)$. There is a natural A_∞-structure on E, and we prove that this structure is mainly determined by the relations of A. In particular, the coefficients of the A_∞-products m_n restricted to the tensor powers of $\mathrm{Ext}^1_A(k_A, k_A)$ give the coefficients of the relations of A. We also relate the m_n's to Massey products.

0. Introduction

The notions of A_∞-algebra and A_∞-space were introduced by Stasheff in the 1960s [1]. Since then, more and more theories involving A_∞-structures (and its cousins, E_∞ and L_∞) have been discovered in several areas of mathematics and physics. Kontsevich's talk [2] at the ICM 1994 on categorical mirror symmetry has had an influence in developing this subject. The use of A_∞-algebras in noncommutative algebra and the representation theory of algebras was introduced by Keller [3–5]. Recently the authors of this paper used the A_∞-structure on the Ext-algebra $\mathrm{Ext}^*_A(k, k)$ to study the non-Koszul Artin-Schelter regular algebras A of global dimension four [6]. The information about the higher multiplications on $\mathrm{Ext}^*_A(k, k)$ is essential and very effective for this work.

Throughout let k be a commutative base field. The definition of an A_∞-algebra will be given in Section 1. Roughly speaking, an A_∞-algebra is a graded vector space E equipped with a sequence of "multiplications" $(m_1, m_2, m_3, \ldots)$: m_1 is a differential, m_2 is the usual product, and the higher m_n's are homotopies which measure the degree of associativity of m_2. An associative algebra E (concentrated in degree 0) is an A_∞-algebra with multiplications $m_n = 0$ for all $n \neq 2$, so sometimes we write an associative algebra as (E, m_2). A differential graded (DG) algebra (E, d) has multiplication m_2 and derivation $m_1 = d$; this makes it into an A_∞-algebra with $m_n = 0$ for $n \geq 3$, and so it could be written as (E, m_1, m_2).

Let A be a connected graded algebra, and let k_A be the right trivial A-module $A/A_{\geq 1}$. The Ext-algebra $\bigoplus_{i\geq 0} \mathrm{Ext}^i_A(k_A, k_A)$ of A is the homology of a DG algebra, and hence by a theorem of Kadeishvili, it is equipped with an A_∞-algebra structure. Note that this A_∞-algebra structure is only unique up to A_∞-isomorphism, not on the nose. We use $\mathrm{Ext}^*_A(k_A, k_A)$ to denote both the usual associative Ext-algebra and the Ext-algebra with any choice for its A_∞-structure. By [7, Ex. 13.4] there is a graded algebra A such that the associative algebra $\mathrm{Ext}^*_A(k_A, k_A)$ does not contain enough information to recover the original algebra A; on the other hand, the information from the A_∞-algebra $\mathrm{Ext}^*_A(k_A, k_A)$ is sufficient to recover A. This is the point of Theorem A, and this process of recovering the algebra from its Ext-algebra is one of the main tools used in [6].

* Corresponding author.
E-mail addresses: dmlu@zju.edu.cn (D.-M. Lu), palmieri@math.washington.edu (J.H. Palmieri), qswu@fudan.edu.cn (Q.-S. Wu), zhang@math.washington.edu (J.J. Zhang).

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We need some notation in order to state the theorem. We say that a graded vector space $V = \bigoplus V_i$ is *locally finite* if each V_i is finite-dimensional. We write the graded k-linear dual of V as $V^{\#}$. As our notation has so far indicated, we use subscripts to indicate the grading on A and related vector spaces. Also, the grading on A induces a bigrading on Ext. We write the usual, homological, grading with superscripts, and the second, induced, grading with subscripts.

Let $\mathfrak{m} = \bigoplus_{i\geq 1} A_i$ be the augmentation ideal of A. Let $Q = \mathfrak{m}/\mathfrak{m}^2$ be the graded vector space of generators of A. The relations in A naturally sit inside the tensor algebra on Q. In Section 4 we choose a vector space embedding of each graded piece A_s into the tensor algebra on Q: a map

$$A_s \hookrightarrow \left(\bigoplus_{m\geq 1} Q^{\otimes m}\right)_s,$$

which splits the multiplication map, and this choice affects how we choose the minimal generating set of relations. See Lemma 5.2 and the surrounding discussion for more details.

Theorem A. *Let A be a connected graded locally finite algebra. Let $Q = \mathfrak{m}/\mathfrak{m}^2$ be the graded vector space of generators of A. Let $R = \bigoplus_{s\geq 2} R_s$ be a minimal graded space of relations of A, with R_s chosen so that*

$$R_s \subset \bigoplus_{1\leq i\leq s-1} Q_i \otimes A_{s-i} \subset \left(\bigoplus_{m\geq 2} Q^{\otimes m}\right)_s.$$

For each $n \geq 2$ and $s \geq 2$, let $i_s : R_s \to (\bigoplus_{m\geq 2} Q^{\otimes m})_s$ be the inclusion map and let i_s^n be the composite

$$R_s \xrightarrow{i_s} \left(\bigoplus_{m\geq 2} Q^{\otimes m}\right)_s \to (Q^{\otimes n})_s.$$

Then there is a choice of A_∞-algebra structures $(m_2, m_3, m_4, \ldots)$ on $E = \mathrm{Ext}_A^*(k_A, k_A)$ so that in any degree $-s$, the multiplication m_n of E restricted to $(E^1)^{\otimes n}_{-s}$ is equal to the map

$$(i_s^n)^{\#} : \left((E^1)^{\otimes n}\right)_{-s} = \left((Q^{\otimes n})_s\right)^{\#} \longrightarrow R_s^{\#} \subset E^2_{-s}.$$

In plain English, the multiplication maps m_n on classes in $\mathrm{Ext}_A^1(k_A, k_A)$ are determined by the relations in the algebra A.

Note that the space Q of generators need not be finite-dimensional—it only has to be finite-dimensional in each grading. Thus the theorem applies to infinitely generated algebras like the Steenrod algebra.

The authors originally announced the result in the following special case; this was used heavily in [6].

Corollary B (*Keller's Higher-Multiplication Theorem in the Connected Graded Case*). *Let A be a connected graded algebra, finitely generated in degree 1. Let $R = \bigoplus_{n\geq 2} R_n$ be a minimal graded space of relations of A, chosen so that $R_n \subset A_1 \otimes A_{n-1} \subset A_1^{\otimes n}$. For each $n \geq 2$, let $i_n : R_n \to A_1^{\otimes n}$ be the inclusion map and let $i_n^{\#}$ be its k-linear dual. Then there is a choice of A_∞-algebra structures $(m_2, m_3, m_4, \ldots)$ on $E = \mathrm{Ext}_A^*(k_A, k_A)$ so that the multiplication map m_n of E restricted to $(E^1)^{\otimes n}$ is equal to the map*

$$i_n^{\#} : (E^1)^{\otimes n} = (A_1^{\#})^{\otimes n} \longrightarrow R_n^{\#} \subset E^2.$$

Keller has the same result for a different class of algebras; indeed, his result was the inspiration for Theorem A. His result applies to algebras the form $k\Delta/I$ where Δ is a finite quiver and I is an admissible ideal of $k\Delta$; this was stated in [8, Proposition 2] without proof. This class of algebras includes those in Corollary B, but since the algebra A in Theorem A need not be finitely generated, that theorem is not a special case of Keller's result. A version of Corollary B was also proved in a recent paper by He and Lu [9] for $\mathbb{N}$-graded algebras $A = A_0 \oplus A_1 \oplus \cdots$ with $A_0 = k^{\oplus n}$ for some $n \geq 1$, and which are finitely generated by $A_0 \oplus A_1$. Their proof was based on the one here (see [9, page 356]).

Here is an outline of the paper. We review the definitions of A_∞-algebras and Adams grading in Section 1. In Section 2 we discuss Kadeishvili's and Merkulov's results about the A_∞-structure on the homology of a DG algebra. In Section 3 we use Merkulov's construction to show that the A_∞-multiplication maps m_n compute Massey products, up to a sign — see Theorem 3.1 and Corollary A.5 for details. In Section 4 the bar construction is described: the dual of the bar construction is a DG algebra whose homology is Ext, and so leads to an A_∞-structure on Ext algebras. Then we give a proof of Theorem A in Section 5, and in Section 6 we give a few examples. Finally, there is an Appendix in which we prove that Merkulov's construction is in some sense ubiquitous among A_∞-algebras; this Appendix was inspired by a comment from the referee.

This paper began as an Appendix in [6].