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Deficiency and the geometric invariants of a group (with an appendix by Pascal Schweitzer)

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Abstract

It is known that the geometric invariants of a group G (which contain information on finiteness properties of certain submonoids and normal subgroups of G) have a description in terms of the vanishing of group homology of G with Novikov-ring-coefficients [see J.-C. Sikorav, Homologie de Novikov associée à une classe de cohomologie réelle de degré un, Thèse Orsay, 1987; R. Bieri, The geometric invariants of a group, in: G.A. Niblo, M.A. Roller (Eds.), Geometric Group Theory, in: London Math. Soc. Lecture Notes Series, vol. 181, Cambridge University Press, Cambridge, 1993; R. Bieri, R. Strebel, Geometric invariants for discrete groups, manuscript-preprint of a monograph (in preparation)], and [R. Bieri, R. Geoghegan, Kernels of actions on non-positively curved spaces, in: P.H. Kropholler, G. Niblo, R. Stöhr (Eds.), Geometry and Cohomology in Group Theory, in: London Math. Soc. Lecture Notes Series, vol. 252, Cambridge University Press, Cambridge, 1998, pp. 24–38]. In a recent paper Kochloukova [D. Kochloukova, Some Novikov rings that are von Neumann finite and knot-like groups (submitted for publication)] uses this to prove a conjecture of E. Rapaport-Strasser on knot-like groups. We extend her approach to establish a rather general relationship between deficiency and the geometric invariants of a group.

MSC: 20J05; 20F65

1. Introduction

1.1. Deficiency

If a group G has a finite presentation in terms of d generators subject to r defining relations then the difference $d - r$ satisfies the inequality $d - r \leq \mathbb{Z}$-rank of G/G'. The maximal value of $d - r$ over all finite presentations of G is the *deficiency* $\operatorname{def}(G)$ of the group G.

1.2. The geometric invariants

Recall that a group (or a monoid) G is of type FP_m, if the trivial G-module $\mathbb{Z}$ admits a $\mathbb{Z}G$-free resolution which is finitely generated in all dimensions $\leq m$. A group G is finitely generated if and only if G is of type FP_1; and G finitely

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presented implies type FP_2, but not vice versa. The geometric invariants $\Sigma^m(G;\mathbb{Z})$, introduced in [4] for $m = 1$ and in [5] for $m > 1$, are defined when G is a group of type FP_m. For simplicity, let us assume that G is finitely generated. By characters of G we mean homomorphisms $\chi : G \to \mathbb{R}$ into the additive group of the real numbers. For each character χ we consider the submonoid of G,

$$G_\chi = \{g \in G \mid \chi(g) \geq 0\},$$

and put

$$\Sigma^m(G;\mathbb{Z}) = \{\chi \mid \text{the submonoid } G_\chi \text{ is of type } FP_m\}.$$

Note that the set of characters is a finite dimensional real vector space and that the geometric invariants form a descending chain of conical subsets

$$\mathrm{Hom}(G,\mathbb{R}) = \mathbb{R}^n = \Sigma^0(G;\mathbb{Z}) \supseteq \Sigma^1(G;\mathbb{Z}) \supseteq \Sigma^2(G;\mathbb{Z}) \supseteq \cdots \supseteq \Sigma^m(G;Z).$$

1.3. The main result

The main result of this note is

Theorem A. *Let G be a finitely presented group.*
(a) *If* $\Sigma^1(G;\mathbb{Z}) \neq \{0\}$ *then* $\mathrm{def}(G) \leq 1$.
(b) *If* $\mathrm{def}(G) = 1$ *then* $\Sigma^1(G;\mathbb{Z}) = \Sigma^2(G;\mathbb{Z})$.
(c) *If* $\mathrm{def}(G) = 1$ *and* $\Sigma^1(G;\mathbb{Z}) \neq \{0\}$ *then G is of cohomological dimension* $\mathrm{cd}(G) \leq 2$.

1.4. Remarks

(a) The first assertion is already contained in the manuscript [6]. The second assertion is new.
(b) The idea to prove the theorem was triggered by Kouchlokova's proof of the Rapaport–Strasser Conjecture on knot-like groups [8,9]. In particular, we are extending and using her result on von Neuman finiteness of the Novikov ring of G with respect to a discrete character $\chi : G \to \mathbb{R}$ i.e., a character whose image is discrete in $\mathbb{R}$.

2. Applications

2.1. An immediate application

An immediate application results, as usually, from the fact that the geometric invariants contain full information on the FP_m-type of normal subgroups with Abelian factor group. In fact, one of the main results of [4] and [5] asserts.

Theorem 1. *Let G be a group of type* FP_m, *k an integer with* $m \geq k \geq 1$, *and N a normal subgroup of G with* G/N *Abelian. Then N is of type* FP_k *if and only if the geometric invariant* $\Sigma^k(G;\mathbb{Z})$ *contains all characters* $\chi : G \to \mathbb{R}$ *with* $\chi(N) = 0$.

Thus, the existence of a finitely generated normal subgroup N with infinite Abelian quotient implies non-vanishing of $\Sigma^1(G;\mathbb{Z})$, and such normal subgroups are of type FP_2, if the first two geometric invariants coincide. Moreover, by [1] normal subgroups N of type FP_2 in finitely generated groups G with $\mathrm{cd}(G) \leq 2$ are known to be free or of finite index. This proves

Corollary B. *Let G be a finitely presented group.*
(a) *If G contains a finitely generated normal subgroup N with* G/N *infinite Abelian then* $\mathrm{def}(G) \leq 1$,
(b) *If* $\mathrm{def}(G) = 1$ *then every finitely generated normal subgroup N with* G/N *Abelian is of type* FP_2; *in fact, if* G/N *is infinite then N is free of finite rank and* $rk_Z(G/N) = 1$.

2.2. Remark

(a) The first assertion of the corollary is contained in the result of Lück [10], asserting the conclusion $\mathrm{def}(G) \leq 1$ even when the assumption "G/N infinite Abelian" is replaced by "G/N contains an element of infinite order". The second assertion seems to be new.