Available online at www.sciencedirect.com
ScienceDirect
Journal of Pure and Applied Algebra 208 (2007) 135–151
JOURNAL OF PURE AND APPLIED ALGEBRA
www.elsevier.com/locate/jpaa

Torsion theories and Galois coverings of topological groups

Marino Gran[a,*], Valentina Rossi[b]

[a] *Lab. Math. Pures et Appliquées, Université du Littoral Côte d'Opale, 50 Rue F. Buisson, 62228 Calais, France*
[b] *Università degli Studi di Udine, Dipartimento di Matematica e Informatica, Via delle Scienze 206, 33100 Udine, Italy*

Received 14 April 2005; received in revised form 11 November 2005
Available online 18 January 2006
Communicated by J. Adámek

Abstract

For any torsion theory in a homological category, one can define a categorical Galois structure and try to describe the corresponding Galois coverings. In this article we provide several characterizations of these coverings for a special class of torsion theories, which we call quasi-hereditary. We describe a new reflective factorization system that is induced by any quasi-hereditary torsion theory. These results are then applied to study various examples of torsion theories in the category of topological groups.

MSC: 18E40; 18C20; 22A99

0. Introduction

The study of semi-abelian categories and of homological categories is becoming a fruitful new subject in categorical algebra (see, for instance, the founding articles [7,22] by Bourn, Janelidze, Márki and Tholen and the reference book [4] by Borceux and Bourn). Among many other things, in these categories it is possible to define a natural notion of torsion theory that extends the classical one introduced by Dickson in any abelian category [14]. The introduction of the notion of torsion theory in this more general context makes it possible to look for new examples of torsion theories in the categories of groups, crossed modules and topological groups.

The idea of studying torsion theories in a homological category was first considered by Bourn and Gran in [10]. In this article some new examples of torsion theories in the categories of topological groups and of topological semi-abelian algebras (introduced by Borceux and Clementino in [5]) were also investigated. This new approach opened the way to other interesting investigations, for instance by Clementino, Dikranjan and Tholen, who further extended the context to the so-called normal categories [13].

In the present paper we study the categorical Galois structure associated with a torsion theory in a homological category, and we provide a complete characterization of the corresponding Galois coverings for a special class of torsion theories, which we call quasi-hereditary. Recall that a torsion theory in a homological category $\mathcal{C}$ is a pair $(\mathcal{T}, \mathcal{X})$ of full replete subcategories of $\mathcal{C}$ such that

* Corresponding author.
E-mail addresses: gran@lmpa.univ-littoral.fr (M. Gran), rossi@dimi.uniud.it (V. Rossi).

0022-4049/$ - see front matter
doi:10.1016/j.jpaa.2005.12.001

(1) the only arrow $f : T \to X$ from $T \in \mathcal{T}$ to $X \in \mathcal{X}$ is the zero arrow 0;
(2) for any object A in $\mathcal{C}$ there exists a short exact sequence

$$0 \longrightarrow T \longrightarrow A \longrightarrow X \longrightarrow 0$$

with $T \in \mathcal{T}$ and $X \in \mathcal{X}$.

We say that a torsion theory $(\mathcal{T}, \mathcal{X})$ is *quasi-hereditary* when its torsion subcategory $\mathcal{T}$ is closed in $\mathcal{C}$ under regular subobjects. Of course, when $\mathcal{C}$ is an abelian category, any quasi-hereditary torsion theory in $\mathcal{C}$ is hereditary, so that the distinction between these two notions vanishes. However, in a general homological category, this distinction is meaningful: for instance, the torsion theory given by the categories (*Grp*(*Ind*), *Grp*(*Haus*)) of indiscrete groups and of Hausdorff groups in the category *Grp*(*Top*) of topological groups is quasi-hereditary, but it is not hereditary.

Any torsion theory gives rise to an admissible Galois structure in the sense of Janelidze [18], and our Theorem 4.5 solves the problem of characterizing the coverings that correspond to a quasi-hereditary torsion theory in a homological category. This result is then used to give a precise description of a corresponding reflective factorization system (Theorem 4.7), which combines the ideas of localization/stabilization of Carboni, Janelidze, Kelly and Paré [12] and of the concordant/dissonant factorization system of Janelidze and Tholen [27].

In the special case of the torsion theory (*Grp*(*Ind*), *Grp*(*Haus*)), we find out that the coverings are exactly the open surjective homomorphisms with a Hausdorff kernel. It is interesting to remark that, in this homological context, there are coverings that are not trivial coverings. The above caracterization is also useful to study another torsion theory in the category of topological groups. The pair of full subcategories (*Grp*(*Conn*), *Grp*(*TotDis*)) of connected groups and of totally disconnected groups is indeed a torsion theory, but it is not a quasi-hereditary torsion theory. For this reason, the characterization of the coverings relative to this second torsion theory is more delicate. An important ingredient helping us to solve this problem comes from a result of Arkhangel'skiĭ [1] in the full subcategory *Grp*(*Haus*) of *Grp*(*Top*): any Hausdorff group is a regular quotient of a totally disconnected group. We can then show that the coverings relative to (*Grp*(*Conn*), *Grp*(*TotDis*)) are exactly the open surjective homomorphisms with a totally disconnected kernel, a notion already considered in the category of connected compact Hausdorff groups by Berestovskij and Plaut [3].

Finally, we would like to mention that our work is related to the work of Janelidze, Márki and Tholen in [23], where these authors solved the problem of classifying the coverings with respect to general radical theories. The main difference with their approach is that, in the present article, we want to get free of the heavy requirement that any object of $\mathcal{C}$ is a regular quotient of an object in $\mathcal{X}$. The present article also provides some new examples of "locally semi-simple coverings" in the non-exact category of topological groups; again, this can be considered as a complement to their work in the exact case (see the last remark in [23]).

The paper is structured as follows:

(1) Effective descent morphisms.
(2) Torsion theories and homological categories.
(3) Quasi-hereditary torsion theories.
(4) Galois structure of torsion theories.
(5) Applications: coverings of topological groups.

In the first section we revise the basic notions of descent theory and prove that the open surjective homomorphisms are effective for descent in any category of topological Maltsev algebras [30,29]. This fact will be useful later on to define a Galois structure in the context of topological algebras. We then recall some properties of homological categories and torsion theories in the second section [8,10]. The third section is devoted to the study of quasi-hereditary torsion theories. We analyse an example of a quasi-hereditary torsion theory which is not hereditary in the category of topological semi-abelian algebras [5]. Necessary and sufficient conditions for a torsion theory to be quasi-hereditary are given in terms of the associated reflection and coreflection. In Section 4 we introduce the Galois structure induced by a torsion theory in a homological category, and we characterize the coverings relative to a quasi-hereditary torsion theory. We analyse the reflective factorization system induced by the coverings. In the last section we consider the coverings corresponding to various torsion theories in the category of topological groups.