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Some results relating to adjacent ideals in dimension two

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Abstract

We answer some of the questions posed by Noh in [S. Noh, Adjacent integrally closed ideals in dimension two, J. Pure Appl. Algebra 85 (2) (1993) 163–184] concerning the existence of adjacent complete ideals in dimension two.

MSC: 13B22; 13H05; 14H50

1. Introduction

The theory of $\mathfrak{m}_O$-primary complete (integrally closed) ideals in a two-dimensional regular local ring $(R, \mathfrak{m}_O)$ was introduced by Zariski and since then, it has been extended by several authors, including Zariski himself, Lipman, Cutkosky and Hoskin [15,10–12, 3–5,9]. This theory appeared as an algebraic counterpart to the classical Italian theory of germs of curves on smooth surfaces developed by Enriques about twenty years before ([6] Book IV, vol. 2). Enriques characterizes the infinitesimal base conditions that can be effectively verified by curves and proposes a procedure (unloading) describing the effective behaviour of curves subjected to non-consistent conditions.

Assuming that R possesses an algebraically closed residue field, Noh studies in [13] the relationship between two complete ideals $J \subset I$ which are *adjacent*, this meaning

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that $l_R(I/J) = 1$. In particular, she shows that every complete $\mathfrak{m}_O$-primary ideal has complete ideals adjacent to it below and above. She also carries out a deep study of the number of integrally closed ideals adjacent to and above a given one. In the last section of [13], Noh poses some questions relating to the existence of complete ideals adjacent to particular ideals of a special kind. In this paper, we answer some of these questions (precisely, questions (3)–(6) of 3.18 in [13]) by making use of the theory of infinitely near points as revised and developed by Casas-Alvero in [2]. We also use some results already obtained in the study of the sandwiched surface singularities and the Nash conjecture of arcs on these singularities [7,8].

2. Preliminaries

Let $(R, \mathfrak{m}_O)$ be a regular local two-dimensional $\mathbb{C}$-algebra and write $S = Spec(R)$. A *cluster* of points of S is a finite set K of points infinitely near or equal to the closed point $O \in S$ such that for any $p \in K$, K contains all points preceding p. By assigning integral multiplicities $\nu = \{\nu_p\}$ to the points of K, we get a *weighted cluster* $\mathcal{K} = (K, \nu)$, the multiplicities ν being called *virtual multiplicities*. If

$$\rho_p^{\mathcal{K}} = \nu_p - \sum_{q \text{ proximate to } p} \nu_q$$

is the *excess* at p of $\mathcal{K}$, *consistent* clusters are those clusters with no negative excesses. *Strictly consistent* clusters are consistent clusters with no points of virtual multiplicity zero.

If $\mathcal{K}$ is any weighted cluster, the set of all equations of curves going through it generates a complete $\mathfrak{m}_O$-primary ideal $H_{\mathcal{K}}$ in R (see [2] 8.3). Two clusters $\mathcal{K}$ and $\mathcal{K}'$ are *equivalent* if $H_{\mathcal{K}} = H_{\mathcal{K}'}$. Any complete $\mathfrak{m}_O$-primary ideal J in R has a weighted cluster of base points $BP(J)$ composed of the points shared by, and the multiplicities of, the curves defined by generic elements of J. Moreover, the maps $J \mapsto BP(J)$ and $\mathcal{K} \mapsto H_{\mathcal{K}}$ are reciprocal isomorphisms between the semigroup of complete $\mathfrak{m}_O$-primary ideals in R and the semigroup of strictly consistent clusters with origin at O. If p is infinitely near or equal to O, we denote by I_p the simple ideal generated by the equations of generic branches going through p, and by $\mathcal{K}(p)$ the weighted cluster corresponding to I_p by the bijection above. In particular, $I = \prod_{i=1}^{n} I_{p_i}^{\alpha_i}$ is the (Zariski) factorization of I if and only if $\mathcal{K} = \sum_{i=1}^{n} \alpha_i \mathcal{K}(p_i)$ (see [2] 8.4 for details). If $\mathcal{K}$ is not consistent, $\widetilde{\mathcal{K}}$ is the cluster given rise to by $\mathcal{K}$ by *unloading* (cf. [2] 4.2 and 4.6). Equivalently, $\widetilde{\mathcal{K}}$ is the unique consistent cluster having the same points as $\mathcal{K}$ and equivalent to it (cf. [2] Section 4.6). The set of points of a cluster equipped with the proximity relations can be encoded in a convenient diagram, called the Enriques diagram of the cluster. Enriques diagrams are explained in [6] Book IV, Chapter 1, and also in [2] Section 3.9.

Following Noh ([13], page 164), two complete ideals $I \supset J$ are said to be *adjacent* if their colengths differ by one, i.e. if $l_R(\frac{I}{J}) = 1$. In this case, we also say that I is *right above* J or that J is *right below* I. Noh proves that given a complete $\mathfrak{m}_O$-primary ideal $I \subset R$, there exist adjacent complete ideals right above and below I (Lemma 1.1 of [13]).

Assume that I is a complete $\mathfrak{m}_O$-primary ideal and write $\mathcal{K} = (K, \nu)$ for the cluster of its base points. Then, the order of I is just the virtual multiplicity of $\mathcal{K}$ at O. If I has order