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Artinian level modules of embedding dimension two

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Abstract

We prove that a sequence of positive integers $(h_0, h_1, \ldots, h_c)$ is the Hilbert function of an artinian level module of embedding dimension two if and only if $h_{i-1} - 2h_i + h_{i+1} \leq 0$ for all $0 \leq i \leq c$, where we assume that $h_{-1} = h_{c+1} = 0$. This generalizes a result already known for artinian level algebras. We provide two proofs, one using a deformation argument, the other a construction with monomial ideals. We also discuss liftings of artinian modules to modules of dimension one.

MSC: 13C05; 13H10; 13D40

1. Introduction

It has been proved (see Iarrobino [8] and Chipalkatti–Geramita [3]) that a sequence of positive integers $(h_0, h_1, \ldots, h_c)$, with $h_0 = 1$, is the Hilbert function of a graded artinian level algebra of embedding dimension two if and only if $h_{i-1} - 2h_i + h_{i+1} \leq 0$ for all $0 \leq i \leq c$, where it is assumed that $h_{-1} = h_{c+1} = 0$. That this condition is necessary follows easily from the condition on the Betti numbers of a level algebra. The sufficiency can be proved by using the Hilbert–Burch theorem, which describes precisely what the free resolutions of graded artinian algebras of embedding dimension two look like.

In this article we will prove that this result generalizes to graded artinian level modules. Level modules were introduced by Boij in [1] as a generalization of level algebras, they are

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graded modules with generators and socle concentrated in single degrees. We will prove that a sequence of positive integers $(h_0, h_1, \ldots, h_c)$ is the Hilbert function of a graded artinian level module of embedding dimension two if and only if

$$h_{i-1} - 2h_i + h_{i+1} \leq 0$$

for all $0 \leq i \leq c$, where we assume that $h_{-1} = h_{c+1} = 0$, and we will call such a sequence a *convex* sequence. That this condition is necessary follows, as in the case of a level algebra, from the condition on the Betti numbers of a level module. We will prove that it is sufficient in two, rather different, ways.

In the first proof (Theorem 3.2 below) we use Macaulay's criterion for modules, which characterizes the possible Hilbert functions of a graded module, to prove that given a convex sequence there is a graded module, level or not, with this sequence as Hilbert function. Then we see that there is a deformation of this module to a level module. When making this deformation we actually work with the dual module, this works since the dual of a level module is a level module. The drawback is that we need to assume that the field we are working over is infinite in order to make the deformation argument work.

The second approach, which leads to Theorem 4.3 below, is more combinatorial in nature and works over any field k. We prove that we may choose monomial ideals I and J in $k[x, y]$ such that $J \subseteq I$ and I/J is an artinian level $k[x, y]$-module with Hilbert function any convex sequence. This also proves the slightly different statement that a sequence of positive integers is the Hilbert function of a multigraded artinian level module if and only if the sequence is convex.

In Section 5 we use a result of Geramita et al. [4] about liftings of monomial ideals. We prove that an artinian quotient of monomial ideals may be lifted to an ideal in the homogeneous coordinate ring of a certain set of reduced points such that the lifted ideal and the homogeneous coordinate ring coincide in high enough degrees. This result holds in any embedding dimension but fits into the context since it can be used to lift the level quotients of monomial ideals constructed in the preceding section.

2. Preliminaries on level modules and dualization

Let $R = k[x_1, x_2, \ldots, x_n]$ be the polynomial ring in n variables over a field k. Consider R as a graded ring by giving each x_i degree one and let $\mathfrak{m} = (x_1, x_2, \ldots, x_n)$ be the unique graded maximal ideal. All R-modules in this article are assumed to be finitely generated and graded. The dth twist of an R-module M, denoted by $M(d)$, is defined by $M(d)_i = M_{i+d}$. If the R-module M has a minimal free resolution given by

$$0 \to \bigoplus_j R(-j)^{\beta_{p,j}} \to \cdots \to \bigoplus_j R(-j)^{\beta_{0,j}} \to M \to 0$$

then $\beta_{i,j}(M) = \beta_{i,j}$ are the graded Betti numbers of M. The graded Betti numbers are independent of the resolution since $\beta_{i,j}(M) = \dim_k \operatorname{Tor}_j^R(M, k)_i$. Denote the Hilbert function of M by $H(M, i) = \dim_k M_i$ and the Hilbert series of M by $H_M(t) = \sum_i H(M, i)t^i$ and recall that $H_M(t)$ may by expressed in terms of the Betti numbers of M