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On the centralizer of the sum of commuting nilpotent elements

George McNinch[1]

Department of Mathematics, Tufts University, 503 Boston Avenue, Medford, MA 02155, USA

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Dedicated to Eric M. Friedlander on his 60th birthday

Abstract

Let X and Y be commuting nilpotent K-endomorphisms of a vector space V, where K is a field of characteristic $p \geqslant 0$. If $F = K(t)$ is the field of rational functions on the projective line $\mathbf{P}^1_{/K}$, consider the $K(t)$-endomorphism $A = X + tY$ of V. If $p = 0$, or if $A^{p-1} = 0$, we show here that X and Y are tangent to the unipotent radical of the centralizer of A in $\mathrm{GL}(V)$. For all geometric points $(a : b)$ of a suitable open subset of $\mathbf{P}^1$, it follows that X and Y are tangent to the unipotent radical of the centralizer of $aX + bY$. This answers a question of J. Pevtsova.

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Let G be a connected and reductive algebraic group defined over an arbitrary field K of characteristic $p \geqslant 0$. Write $\mathfrak{g} = \mathrm{Lie}(G)$, and consider the extension field $F = K(t)$ with t transcendental over K. For convenience, we fix an algebraically closed field k containing both K and t.

If $X, Y \in \mathfrak{g}(K)$ are nilpotent and $[X, Y] = 0$, then $A = X + tY \in \mathfrak{g}(F)$ is again nilpotent. Write C for the centralizer of A in G, and write $R_u C$ for the unipotent radical of C. Under favorable restrictions on the characteristic, the groups C and $R_u C$ are defined over $K(t)$. In this note, I want to answer—at least in part—a question put to me by Julia Pevtsova at the

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E-mail address: george.mcninch@tufts.edu.

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July 2004 meeting in Snowbird, Utah. With notation as before, this question may be stated as follows:

Question 1. When is it true that $X, Y \in \operatorname{Lie} R_u C$?

To begin the investigation, the first section of the paper includes some elementary results concerning G-varieties in case the algebraic group G acts with a finite number of orbits. For the most part, the use of these results could be avoided in the present application, but there is perhaps some interest in recording them.

After these preliminaries, I am mainly going to investigate Question 1 in case the K-group is $G = \mathrm{GL}(V)$, where V is a finite dimensional k-vector space defined over K; this means there is a given K-subspace $V(K)$ for which the inclusion induces an isomorphism $V(K) \otimes_K k \simeq V$.

Section 2 contains well-known material on nilpotent orbits, mainly for the group $\mathrm{GL}(V)$; this material is used in Section 3 where we prove our main result (Theorem 21) giving a partial answer to Question 1 when G is the group $\mathrm{GL}(V)$. Section 4 contains some remarks about more general semisimple groups.

Let me make a few remarks about possible reasons for interest in the main result of this paper. Pevtsova's interest concerns finite group schemes over a field K of characteristic $p > 0$; see e.g. [2]. Basic but important examples are the commutative, étale, unipotent group schemes; consider e.g. a constant finite group scheme E which "is" an elementary Abelian p-group. If (ρ, M) is a K-representation of E, the matrices $1 - \rho(g) = \rho(1 - g) \in \operatorname{End}_K(M)$ are nilpotent for $g \in E$. More generally, if x is in the augmentation ideal of the group algebra KE, then $\rho(x)$ is nilpotent, and is a linear combination of commuting nilpotent matrices $\rho(1 - g)$ for various elements $1 \neq g$ of E. Pevtsova's question was aimed at understanding properties of the Jordan block structure of suitably generic such x.

In a somewhat different direction, if G is a reductive group over K and $\mathcal{N} \subset \mathfrak{g}$ denotes the variety of nilpotent elements, one is interested in studying the subvariety $\mathcal{V}_2 \subset \mathcal{N} \times \mathcal{N}$ of commuting pairs:

$$\mathcal{V}_2 = \{(X_1, X_2) \in \mathcal{N}^2 \mid [X_1, X_2] = 0\};$$

see e.g. [11]. Any K-point

$$x = (X_1, X_2) \in \mathcal{V}_2(K)$$

determines a nilpotent element $A = X_1 + t X_2 \in \mathfrak{g}(F)$ with $F = K(t)$ as before. One might hope to exploit the results of this paper to study properties of the variety $\mathcal{V}_2$.

1. Groups acting with finitely many orbits

In this section, we work "geometrically"—i.e. over the algebraically closed field k. The results recorded here are elementary and without doubt are well-known; however, I do not know of an adequate reference.