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Realizing modules over the homology of a DGA

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Abstract

Let A be a DGA over a field and X a module over $H_*(A)$. Fix an A_∞-structure on $H_*(A)$ making it quasi-isomorphic to A. We construct an equivalence of categories between A_{n+1}-module structures on X and length n Postnikov systems in the derived category of A-modules based on the bar resolution of X. This implies that quasi-isomorphism classes of A_n-structures on X are in bijective correspondence with weak equivalence classes of rigidifications of the first n terms of the bar resolution of X to a complex of A-modules. The above equivalences of categories are compatible for different values of n. This implies that two obstruction theories for realizing X as the homology of an A-module coincide.

MSC: 55S35; 55U15; 16E45

1. Introduction

Let A be a differential graded algebra over a field k and let $R = H_*(A)$ be its homology. We say that an R-module X is *realizable* if there exists a differential graded module M over A with $H_*(M) \simeq X$. This paper deals with two obstruction theories for answering the question of whether or not a module is realizable.

One obstruction theory is based on the theory of A_n-structures. In [10], Stasheff introduced a hierarchy of higher homotopy associativity conditions for multiplications on chain complexes. An A_2-structure is just a bilinear multiplication m_2, while an A_3-structure is an A_2-structure together with a homotopy m_3 between the two ways of bracketing a 3-fold product. An A_∞-structure consists of a sequence of higher associating homotopies m_n satisfying certain conditions (see Section 2 for the definitions and also [6] for an excellent introduction to the theory of A_∞-algebras and modules).

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Kadeishvili proved [5] that there is a an A_∞-structure on $H_*(A)$ making it quasi-isomorphic to A as an A_∞-algebra. Such an equivalence induces an equivalence of derived categories of A_∞-modules and the derived category of (homologically unital) A_∞-modules over A is equivalent to the usual derived category of DG modules over A. This implies that a module X is realizable if and only if it admits the structure of an A_∞-module over $H_*(A)$. The $H_*(A)$-module structure on X makes it an A_2-module over the A_∞-algebra $H_*(A)$ and so the problem of realizability is naturally broken down into the problem of extending an A_n-module structure on X to an A_{n+1}-structure for successive n.

Given an A_n-structure on X, the obstruction to extending the underlying A_{n-1}-structure to an A_{n+1}-structure lies in $\mathrm{Ext}^{n,n-2}(X, X)$ (see Proposition 3.4). The original motivation for this paper was the observation that this first obstruction, i.e. the obstruction to extending the given A_2-structure to an A_4-structure, is the primary obstruction to realizability described by other means in a recent paper of Benson, Krause and Schwede [1].

In [1, Appendix A], the authors describe a general obstruction theory for realizability based on the notion of a Postnikov system (see Definition 5.1). This approach has its roots in stable homotopy theory (see [12] for example). The basic idea is the following. Since free modules and maps between them are clearly realizable, we can realize a free resolution for X in the derived category of A-modules. The problem is then whether this "chain complex of modules up to homotopy" can be rigidified to an actual chain complex of A-modules. This is explained in detail in Appendix A (see also [1]). The Postnikov system approach can be applied more generally [1, Appendix A] to the problem of realizing modules over endomorphism rings of compact objects in triangulated categories.

Benson, Krause and Schwede define a canonical Hochschild class[2] $\gamma_A \in HH^{3,1}(H_*(A))$ and show that the primary obstruction to building a Postnikov system for X is the cup product $\mathrm{id}_X \cup \gamma_A \in \mathrm{Ext}^{3,1}(X, X)$. Looking more closely one sees that the cocycles representing γ_A are (up to sign) precisely the A_3-structures on $H_*(A)$ which extend to an A_∞-structure quasi-isomorphic to A. It turns out that the primary obstruction $\mathrm{id}_X \cup \gamma_A$ to realizing X is the obstruction to putting an A_4-module structure on X (over any of these quasi-isomorphic A_∞-structures on $H_*(A)$) and so the primary obstructions to realizing a module coincide from the two points of view. A natural question is then whether the two obstruction theories described above coincide in general. We answer this question in the affirmative.

We construct a functor from A_n-modules over $H_*(A)$ to filtered differential graded A-modules. This filtration gives rise to an $(n-1)$-Postnikov system for X and our main result is that this functor induces an equivalence of categories.

Theorem 1.1. *Let A be a differential graded algebra over a field. There is an equivalence of categories between minimal A_n-modules over $H_*(A)$ and $(n-1)$-Postnikov systems based on the bar resolution.*[3]

This is proved below as Corollary 5.6 and Theorem 5.8. It implies in particular that for a fixed $H_*(A)$-module X, the moduli groupoid of A_n-structures on X is equivalent to the groupoid of $(n-1)$-Postnikov systems based on the bar resolution for X with isomorphisms which are the identity on the bar resolution (see Corollary 5.10). These equivalences of categories are compatible with the forgetful functors for varying n and hence yield an equivalence of the two obstruction theories (see Theorem 5.5).

It is a folk theorem in homotopy theory that given a chain complex $B_\bullet$ in a homotopy category $\mathrm{Ho}(\mathcal{C})$, n-Postnikov systems based on $B_\bullet$ correspond to rigidifications of the first n-terms of $B_\bullet$ to a chain complex in $\mathcal{C}$. We explain this in Appendix A. A consequence of this is the following result which is a direct consequence of Corollary 5.6 and Proposition A.6.

Corollary 1.2. *Isomorphism classes of A_n-module structures on X are in bijective correspondence with weak equivalence classes of rigidifications of the complex formed by the first n terms of the bar resolution for X.*

It would be interesting to know to what extent the previous results generalize to an abstract stable homotopy theoretic context.

The equivalence of Theorem 1.1 is reminiscent of the equivalence originally established by Stasheff [10] between the existence of an A_n-structure (defined as a certain diagram in the homotopy category) and an A_n-form on a topological space X. It would be interesting to understand the precise relation.

[2] The class in [1] is actually in $HH^{3,-1}(H^*(A))$. This is because of our convention (which follows [11]) that the kth shift $X[k]$ is the kth desuspension of X in the derived category, while in [1] it is the kth suspension.

[3] See Definition 5.1.