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Influence of permutizers of subgroups on the structure of finite groups☆

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ABSTRACT

A group G is said to satisfy the maximal permutizer condition if the permutizer of any maximal subgroup M of G in G, $P_G(M)$, is G. In this paper, we characterize the supersolubility of finite groups by using the maximal permutizer condition. We also get some results for when both G/N and N are supersoluble, which implies that G is supersoluble. Our results unify or generalize some known results.

1. Introduction

A normalizer of a subgroup plays an important role in the study of finite groups. We say a group satisfies the normalizer condition if every proper subgroup of a group is strictly smaller than its normalizer. An important and classic result is that a finite group satisfies the normalizer condition if and only if it is a finite nilpotent group. We also know that G is a finite nilpotent group if and only if every maximal subgroup M of G is normal in G. We define the maximal normalizer condition as the normalizer of every maximal subgroup M of G that is equal to G. Then we know that the normalizer and the maximal normalizer condition are equivalent. This tells us that the normalizer condition is a strong condition.

A natural way to generalize the normalizer condition is to replace the normalizer of a subgroup by its permutizer. The permutizer of a subgroup H of G is defined to be the subgroup generated by all cyclic subgroups of G that permute with H, i.e. $\langle x \in G | \langle x\rangle H = H\langle x\rangle\rangle$, denoted by $P_G(H)$.

In [3], Beidleman and Robinson defined the permutizer condition. A group is said to satisfy the permutizer condition if every proper subgroup of a group is strictly smaller than its permutizer.

Some authors have obtained some interesting results about the permutizer condition.

Zhang came to the following conclusions in [7]:

(a) If G is a finite soluble group and satisfies the permutizer condition, then
  (1) G is supersoluble if and only if no quotient group of G is isomorphic to S_4.
  (2) for any odd prime p, G is p-supersoluble.
  (3) $Q \in \mathrm{Syl}_2(G')$, then $Q \trianglelefteq G$, and G/Q is supersoluble.

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In [3], the authors proved an important result.

(b) Each finite group G satisfying the permutizer condition is soluable and every chief factor of G has order 4 or a prime.

Later, Xiaolei Liu and Yanming Wang [5] weakened the conditions of (a) and (b); by only considering the permutizers of almost maximal subgroups, they got the following results:

(1) Suppose $P_G(M) = G$ for any maximal subgroup M of finite group G, then (a) holds.

(2) Suppose $P_G(M) > M$ for any almost maximal subgroup M of G, every chief factor of G has order 4 or a prime.

A proper subgroup M is called an almost maximal subgroup of G if M is a maximal subgroup or $|G : M|$ is a power of a prime number.

Motivated by the above research, we aim to give some new results on the permutizers of subgroups; some of these unify or generalize the above results. As a by product, we also get some results about when G/N and N are supersoluble implying that G is supersoluble.

All groups considered in this paper are finite. G always denotes a finite group, p a prime, $\pi(G)$ the set of prime divisors of the order of group G, S_n the symmetric group of degree n, A_n the alternating group of degree n and M_G the core of M in G.

2. Preliminaries

Definition 2.1. A group G is said to satisfy the permutizer condition in G if $P_G(H)$ strictly contains H for any subgroup H of G.

Definition 2.2. A group G is said to satisfy the maximal permutizer condition if $P_G(M) = G$ for any maximal subgroup M of G.

Definition 2.3. Let S be a group. A group G is called S-free if no quotient group of any subgroup of G is isomorphic to S.

Lemma 2.4. *Let H be a subgroup of G and N a normal subgroup of G. Then:*

(1) $N_G(H) \le P_G(H)$.

(2) $P_{G/N}(HN/N) \ge P_G(H)N/N$.

(3) If N is also a subgroup of H, then $P_{G/N}(H/N) = P_G(H)/N$.

Proof. (1) This is evident.

(2) Since $P_G(H) = \langle x | x \in G, \langle x\rangle H = H\langle x\rangle\rangle$, $P_G(H)N/N = \langle xN | x \in G, \langle x\rangle H = H\langle x\rangle\rangle$. Let x be an element of G such that $H\langle x\rangle = \langle x\rangle H$. Then $xN \in P_G(H)N/N$. We have $(HN/N)(\langle x\rangle N/N) = HN\langle x\rangle N/N = H\langle x\rangle N/N = \langle x\rangle HN/N = (\langle x\rangle N/N)(HN/N)$, which implies $xN \in P_{G/N}(HN/N)$. Hence $P_{G/N}(HN/N) \ge P_G(H)N/N$.

(3) By (2), it suffices to prove $P_{G/N}(H/N) \le P_G(H)/N$.

Let $xN \in P_{G/N}(H/N)$ such that $(H/N)\langle xN\rangle = \langle xN\rangle(H/N)$. Since $N \le H$, it follows that $(\langle x\rangle H)/N = (H\langle x\rangle)/N$. Hence $\langle x\rangle H = H\langle x\rangle$. □

Lemma 2.5 ([3, Lemma 3.2]). *Let P be a p-group and let N be a nontrivial, elementary abelian normal subgroup of P which has a complement X in P. If $P = \langle y\rangle X$ for some element y, then $|N| = p$ if $p > 2$ and $|N| \le 4$ if $p = 2$.*

Lemma 2.6. *G is soluble if G satisfies the maximal permutizer condition.*

Proof. Let M be a maximal subgroup of G and $P_G(M) = G$. Then there exists an element $x \in G \setminus M$ such that $G = \langle x\rangle M = M\langle x\rangle$, which implies that M has a cyclic supplement in G. By [1, Theorem 1.1], G is soluble. □

Lemma 2.7 ([4, VI, Theorem 4.7]). *Suppose $G = G_1G_2$. For any prime p, there exist P, P_1, P_2 such that $P = P_1P_2$, where $P \in \mathrm{Syl}_p(G)$, $P_i \in \mathrm{Syl}_p(G_i)$, $i = 1, 2$.*

Lemma 2.8. *Let H be a subgroup of G such that $|G : H|$ is a π-number. If there is a nilpotent subgroup K of G such that $G = HK$, then $G = HK_\pi$, where K_π is a π-Hall subgroup of K.*

Proof. Let $K = K_{\pi'}K_\pi$, where $K_{\pi'}$ is a π'-Hall subgroup of K. We can assume that $K_{\pi'} > 1$. Let K_p be a nonidentity Sylow p-subgroup of $K_{\pi'}$. Since $|G : H|$ is a π-number, Sylow p-subgroups of H are Sylow p-subgroups of G. On the other hand, by Lemma 2.7, there exists a Sylow p-subgroup H_p of H such that H_pK_p is a Sylow p-subgroup of G. So $H_pK_p = H_p$, which implies $K_p \le H_p \le H$. Thus we have $K_{\pi'} \le H$. Hence $G = HK = H(K_{\pi'}K_\pi) = (HK_{\pi'})K_\pi = HK_\pi$. □

Lemma 2.9 ([6, IX, Theorem 9.3.3]). *Suppose $1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_s = G$ is a chief series of G. Then $F_p = \bigcap_{p||G_{i+1}/G_i|} C_G(G_{i+1}/G_i)$ is p-nilpotent, and $F_p(G)$ contains all normal p-nilpotent subgroups of G.*

Lemma 2.10 ([2, Proposition 2.3]). *Let G be a group and N a normal subgroup of G such that $G = HN$ for some subgroup H of G. Suppose M is a maximal subgroup of G with $N \le M$. Then $H \cap M$ is a maximal subgroup of H.*