Available online at www.sciencedirect.com
ScienceDirect
Journal of Pure and Applied Algebra 212 (2008) 349–363
JOURNAL OF PURE AND APPLIED ALGEBRA
www.elsevier.com/locate/jpaa

A tropical approach to secant dimensions

Jan Draisma

Department of Mathematics and Computer Science, Technische Universiteit Eindhoven, P.O. Box 513, 5600 MB Eindhoven, The Netherlands

Received 14 February 2006; received in revised form 7 May 2007; accepted 10 May 2007
Available online 24 June 2007

Communicated by A.V. Geramita

Abstract

Tropical geometry yields good lower bounds, in terms of certain combinatorial–polyhedral optimisation problems, on the dimensions of secant varieties. The approach is especially successful for toric varieties such as Segre–Veronese embeddings. In particular, it gives an attractive pictorial proof of the theorem of Hirschowitz that all Veronese embeddings of the projective plane except for the quadratic one and the quartic one are non-defective; and indeed, no Segre–Veronese embeddings are known where the tropical lower bound does not give the correct dimension. Short self-contained introductions to secant varieties and the required tropical geometry are included.

MSC: 14N05; 14M25; 52B55

1. Introduction

Secant varieties are rather classical objects of study in algebraic geometry: given a closed subvariety X of some projective space $\mathbb{P}^m$, and given a natural number k, one tries to describe the union of all subspaces of $\mathbb{P}^m$ that are spanned by k points on X. We call the Zariski closure of this union the kth *secant variety* of X, and denote it by kX. To avoid confusion: some authors call this the $(k-1)$st secant variety. So in this paper $2X$ is the variety of secant lines, traditionally called *the* secant variety of X. We will refer to all kX as (higher) secant varieties, and to their dimensions as (higher) secant dimensions. The standard reference for secant varieties is [32].

Already the most basic of all questions about the secant varieties of X poses unexpected challenges, namely: what are their dimensions? This question is of particular interest when X is a *minimal orbit* in a representation space of a reductive group. These minimal orbits comprise Segre embeddings of products of projective spaces, Plücker embeddings of Grassmannians, and Veronese embeddings of projective spaces; see Section 6. Among these instances, only the secant dimensions of the Veronese embeddings are completely known, from the ground-breaking work of Alexander and Hirschowitz [1–3,20]; see also the recent preprint [10] which simplifies the proof for cubics. Secant dimensions of Segre powers of the projective line are almost entirely known [12].

This paper introduces a new approach to secant dimensions, based on tropical geometry. Tropical geometry is a tool for transforming algebraic–geometric questions into polyhedral–combinatorial ones. Recommended references are

E-mail address: j.draisma@tue.nl.

0022-4049/$ - see front matter
doi:10.1016/j.jpaa.2007.05.022

[7,18,24,26–28] and the references therein—however, all background in tropical geometry needed here is reviewed in Section 4.

In Sections 2 and 3 I present the tropical lower bounds on secant dimensions in terms of certain polyhedral optimisation problems. After a review of the necessary tropical geometry in Section 4 we prove the lower bounds in Section 5. In Section 6 I recall the notion of minimal orbits, and give two lower bounds on their secant dimensions. One of them is well known in special cases; for instance, it uses rook coverings for Segre varieties, and a variation of these for Grassmannians [11,13,17,30]. The other seems to be good for Segre products of Veronese embeddings.

Then in Section 7 we apply the tropical lower bounds to Segre varieties, Veronese embeddings, and Grassmannians, and show that they are better than the bounds from Section 6. As an example, we re-prove the theorem that all but two Veronese embeddings of the projective plane are non-defective; this was proved earlier by Hirschowitz [20] using his "Horace method" and by Miranda and Dumitrescu using degenerations (private communication). Also, I give a nice proof that the 6-fold Segre power of the projective line is non-defective; this is the first case not covered by [12]. Finally, Seth Sullivant and Bernd Sturmfels pointed out the paper [16] to me, in which tropical secant varieties of ordinary linear spaces are considered. In the case of toric varieties, where in my approach a monomial parametrisation can be used, the polyhedral set Q appearing in Theorem 2.1 is equal to a tropical secant variety of an ordinary linear space of the type that Develin deals with, as the referee pointed out to me. Develin, however, focuses more on its combinatorial structure than its dimension.

In conclusion, the tropical approach is conceptually very simple, but shows very promising results when tested on concrete examples. However, it also raises many intriguing combinatorial–polyhedral optimisation problems; I do not know of any efficient algorithms solving these.

2. Joins, secant varieties, and first results

Rather than projective varieties, we consider closed cones in affine spaces. So let K be an algebraically closed field of characteristic 0, let V be a finite-dimensional vector space over K, and let C, D be closed cones: Zariski-closed subsets of V that are closed under scalar multiplication. Then we define the *join* of C and D as

$$C + D := \overline{\{c + d \mid c \in C, d \in D\}}.$$

Note that in taking the closure we ignore the subtle question of which elements of $C + D$ can actually be written as $c + d$ with $c \in C$ and $d \in D$; in this paper we are only interested in dimensions. There is an obvious upper bound on the dimension of $C + D$, namely $\min\{\dim C + \dim D, \dim V\}$—indeed, the summation map $C \times D \to C + D$ is dominant. We call this upper bound the *expected dimension* of $C + D$. If $C + D$ has strictly lower dimension than expected, then we call $C + D$ *defective*; otherwise, we call $C + D$ *non-defective*.

Taking the join is an associative (and commutative) operation on closed cones in V, so given k closed cones $C_1, \ldots, C_k$, their join $C_1 + \cdots + C_k$ is well-defined. Again, we call this join defective or non-defective according as its dimension is smaller than or equal to $\min\{\dim V, \sum_i \dim C_i\}$.

In particular, taking all C_i equal to a single closed cone C we obtain kC, called the kth *secant variety* of C. We call C defective if and only if kC is defective for some $k \geq 0$, and we call the numbers $\dim kC,\ k \in \mathbb{N}$ the *secant dimensions* of C. The standard reference for joins and secant varieties is [32].

Typically, one considers a class of cones (e.g., the cones over Grassmannians), one knows a short explicit list of defective secant varieties of cones in this class, and wishes to prove that all other secant varieties of cones in this class are non-defective. One then needs *lower bounds* on secant dimensions that are in fact *equal* to the expected dimensions—so that one can conclude equality.

Our approach towards such lower bounds focuses on the following, special situation: suppose that $C_1, \ldots, C_k$ are closed cones in V, and single out a basis $e_1, \ldots, e_n$ of V. The method depends on this basis, but in our applications there will be natural bases to work with. Let $y_1, \ldots, y_n$ be the dual basis of V^*. Assume for simplicity that none of the C_i is contained in any coordinate hyperplane $\{y_b = 0\}$. Furthermore, suppose that for each i we have a finite-dimensional vector space V_i over K, again with a fixed basis $x = (x_1, \ldots, x_{m_i})$ of V_i^*, and a polynomial map $f_i : V_i \to V$ that maps V_i dominantly into C_i. In particular, every C_i is irreducible.

Write each f_i, relative to the bases of V_i and V, as a list $(f_{i,b})_{b=1}^n$ of polynomials $f_{i,b} \in K[x_1, \ldots, x_{m_i}]$; the fact that we use the same letter x to indicate coordinates on the distinct V_i will not lead to any confusion. For every