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ScienceDirect
Journal of Pure and Applied Algebra 211 (2007) 15–23
JOURNAL OF PURE AND APPLIED ALGEBRA
www.elsevier.com/locate/jpaa

Syzygies of projective bundles

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Received 31 July 2006; received in revised form 15 November 2006
Available online 4 January 2007
Communicated by A.V. Geramita

Abstract

In this paper we study defining equations and syzygies among them of projective bundles. We prove that for a given $p \geq 0$, if a vector bundle on a smooth complex projective variety is sufficiently ample, then the embedding given by the tautological line bundle satisfies property N_p.

MSC: 14N05; 13D02

1. Introduction

The goal of this article is to study the defining equations and their syzygies of projective bundles. Let Y be a smooth complex projective variety of dimension $n \geq 1$ and let $\mathcal{E}$ be a vector bundle of rank r on Y. Let

$$X = \mathbb{P}_Y(\mathcal{E})$$

be the associated projective bundle with the tautological line bundle $L = \mathcal{O}_{\mathbb{P}_Y(\mathcal{E})}(1)$ and the projection morphism $\pi : X \to Y$. It is a well-known fact that if $\mathcal{E}$ is a sufficiently ample vector bundle, then L is very ample. In this case, the complete linear series of L defines an embedding

$$\phi_L : X \hookrightarrow \mathbb{P}H^0(X, L) = \mathbb{P}^r$$

such that each fiber of π is embedded as a linear subspace of $\mathbb{P}^r$. Once we know that L is very ample, it is natural to study the defining equations and syzygies among them of $\phi_L(X) \subset \mathbb{P}^r$. Along this line, we are interested in the condition that (X, L) satisfies the property N_p which was defined by M. Green and R. Lazarsfeld. For a precise statement, let S be the homogeneous coordinate ring of $\mathbb{P}^r$ and let

$$R(L) = \bigoplus_{\ell \in \mathbb{Z}} H^0(X, L^\ell)$$

be the graded ring of twisted global sections of L. Then $R(L)$ is a finitely generated graded S-module, so it has a minimal graded free resolution. We say that (X, L) satisfies property N_p if $R(L)$ admits the minimal free resolution

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0022-4049/$ - see front matter
doi:10.1016/j.jpaa.2006.12.010

of the form

$$\cdots \to S^{\beta_{p,1}}(-p-1) \to \cdots \to S^{\beta_{2,1}}(-3) \to S^{\beta_{1,1}}(-2) \to S \to R(L) \to 0.$$

Therefore property N_0 holds if and only if $\phi_L(X) \subset \mathbb{P}^r$ is a projectively normal variety, property N_1 holds if and only if property N_0 is satisfied and the homogeneous ideal of $\phi_L(X)$ is generated by quadrics, and property N_p holds for $p \geq 2$ if and only if it has property N_1 and the kth syzygies among the quadrics are generated by linear syzygies for all $1 \leq k \leq p-1$.

When $\mathcal{E}$ is a line bundle and hence $X = Y$, the guiding principle is that if L is more and more ample, then the minimal free resolution of the embedding defined by L has more and more long linear steps. See the following results:

Theorem 1.1. *Let X be a projective variety of dimension n.*

(1) *(Inamdar, [3]) Let $A \in \mathrm{Pic}X$ be an ample line bundle. Then for every positive integer p, there exists a number m_p such that mA has property N_p if $m \geq m_p$.*
(2) *(Ein and Lazarsfeld, [1]) Assume that X is smooth and A is a very ample line bundle on X.*
   (i) *If $D \in \mathrm{Pic}X$ is nef, then $K_X + fA + D$ satisfies property N_p for $f \geq n+1+p$.*
   (ii) *If the degree of A is d, then ℓA satisfies property $N_{\ell+1-d}$.*

In this paper we generalize Theorem 1.1 to projective bundles over an arbitrary smooth complex projective variety.

Theorem 1.2. *Let $\mathcal{E}$ be a nef vector bundle of rank r on a smooth complex projective variety Y of dimension n. Let $X = \mathbb{P}_Y(\mathcal{E})$ be the associated projective bundle with the tautological line bundle L and the projection morphism $\pi : X \to Y$. For $A, D \in \mathrm{Pic}Y$, suppose that A is very ample and D is nef. Let e be an integer such that $A^e \otimes \mathcal{E} \otimes \mathcal{E}^*$ is a nef vector bundle. Then*

(1) *For $f \geq er+n+1+p$, $L + \pi^*(K_Y + fA + D)$ satisfies property N_p.*
(2) *If d is the degree of A, then $L + \pi^* fA$ satisfies property N_p for all $f \geq er+d-1+p$.*

Clearly the integer e depends only on X and A since

$$(\mathcal{E} \otimes B) \otimes (\mathcal{E} \otimes B)^* = \mathcal{E} \otimes \mathcal{E}^*$$

for any $B \in \mathrm{Pic}Y$. Since $H + \pi^* B$ is the tautological line bundle of the projective bundle $\mathbb{P}_Y(\mathcal{E} \otimes B)$, Theorem 1.2 means that if a vector bundle is more and more positive, then the minimal free resolution of the homogeneous ideal of the projective bundle embedded by the tautological line bundle has more and more long linear steps. Also if $\mathcal{E} = \mathcal{O}_Y^{\oplus r}$, then $X = Y \times \mathbb{P}^{r-1}$ and $e = 0$. Therefore $L + \pi^*(K_Y + fA + B)$ satisfies property N_p if $f \geq n+1+p$. For the proof of Theorem 1.2, recall that property N_p of a very ample line bundle is governed by several vanishing of cohomology groups on the variety by Green's work [2]. In our case, there is the projection morphism $\pi : X \to Y$. And our main idea is that for the projective bundle X, one can reduce the desired vanishing of cohomology groups on X to those on Y. For details, see Lemma 3.1 and Proposition 3.2. Then we use Ein–Lazarsfeld's method in [1] to obtain the desired vanishing of cohomology groups on Y. To this end, we need to show a variant of vanishing theorems for ample vector bundles on Y. See Section 2.4.

Remark 1.1. For $a \geq 2$, the properties N_p of $(X, aL + \pi^* B)$ are closely related to those of the a-uple Veronese embedding of $\pi^{-1}(P) \cong \mathbb{P}^{r-1}$ for $P \in Y$. For details, see Section 3 in [8]. For an example, if $r \geq 3$ and $a \geq 3$, then $(X, aL + \pi^* B)$ fails to satisfy property N_{3a-2} no matter how ample $B \in \mathrm{Pic}Y$ is. □

2. Preliminaries

2.1. Castelnuovo–Mumford regularity

Let X be a smooth projective variety of dimension n and let $B \in \mathrm{Pic}X$ be an ample line bundle which is globally generated. A coherent sheaf $\mathcal{F}$ on X is said to be m-regular with respect to B if

$$H^i(X, \mathcal{F} \otimes B^{m-i}) = 0 \quad \text{for all } i \geq 1.$$