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On the classification problem of the quasi-isomorphism classes of free chain algebras

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Abstract

The present paper is devoted to the classification problem of the quasi-isomorphism classes of free differential graded algebras (dgas) over a (P.I.D) R. We introduce the notion of coherent homomorphisms, perfect and quasi-perfect dgas (the Adams–Hilton model of simply connected CW-complex such that $H_*(X, R)$ is free is a such a dga) and our first main result asserts that two perfect (quasi-perfect) dgas are quasi-isomorphic if and only if their Whitehead exact sequences are coherently isomorphic. Moreover we define the notion of a strong isomorphism between the Whitehead exact sequences and we show that two free R-dgas, of which their Whitehead exact sequences are strongly isomorphic, are quasi-isomorphic.

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1. Introduction

Let R be a principal ideal domain. To each free differential graded algebra (free dga for short) $(T(V), \partial)$, where $V = V_{\geq 1}$, is associated a long exact sequence, called the Whitehead exact sequence:

$$\cdots \to H_{n+1}(V,d) \xrightarrow{b_{n+1}} \Gamma_n^{T(V)} \to H_n(T(V)) \to H_n(V,d) \xrightarrow{b_n} \cdots \to H_3(V,d)$$

$$\xrightarrow{b_3} \Gamma_2^{T(V)} = H_1(V,d) \otimes H_1(V,d) \to H_2(T(V)) \twoheadrightarrow H_2(V,d)$$

where $\Gamma_n^{T(V)} = \ker\ (H_n(T(V_{\leq n})) \to V_n)$ and where (V, d) is the chain complex of the indecomposables of $(T(V), \partial)$.

Originally this sequence was introduced by Whitehead in [8] in order to classify the homotopy types of simply connected CW-complexes of dimension 4 and later Baues [2] constructed this sequence for free dgas and he proved

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that two free dgas $(T(V), \partial)$ and $(T(W), \delta)$ such that $H_i(V, d) = H_i(W, d') = 0$, for all $i \geq 4$, are quasi-isomorphic if and only if their Whitehead exact sequences are isomorphic in the following sense given by Baues: there exist isomorphisms $f_i : H_i(V, d) \to H_i(W, d')$, $1 \leq i \leq 3$ and h_2 making the following diagram commutes:

$$\begin{array}{ccccccc} H_3(V,d) & \xrightarrow{b_3} & H_1(V,d)\otimes H_1(V,d) & \to & H_2(T(V)) & \twoheadrightarrow & H_2(V,d) \\ \downarrow f_3 & & \downarrow f_1\otimes f_1 & & \downarrow h_2 & & \downarrow f_2 \\ H_3(W,d') & \xrightarrow{b'_3} & H_1(W,d')\otimes H_1(W,d') & \to & H_2(T(W)) & \twoheadrightarrow & H_2(W,d') \end{array}$$

Recall that a dga morphism $\alpha : (T(V), \partial) \to (T(W), \delta)$ is called a quasi-isomorphism if α induces an isomorphism in homology. In this case we say that $(T(V), \partial)$ and $(T(W), \delta)$ are quasi-isomorphic.

The Whitehead exact sequence does not determine the quasi-isomorphism class in general. So three natural questions arise from this observation. They can be formulated as follows:

1. Can one define a notion of homomorphism between two Whitehead exact sequences associated with two given free dgas generalizing the notion given by Baues?
2. For which class of free dgas do the Whitehead exact sequences determine the quasi-isomorphism classes?
3. Let $(T(V), \partial)$ and $(T(W), \delta)$ be two free dgas such that their Whitehead exact sequences are isomorphic in the sense of question 1. What condition should we add to have a quasi-isomorphism between $(T(V), \partial)$ and $(T(W), \delta)$?

This paper is devoted to answering these questions. For the first and second one we introduce the notion of coherent homomorphisms between Whitehead exact sequences and we define two new classes of free dgas called 'perfect' and 'quasi-perfect'. These classes contain all free dgas $(T(V), \partial)$ such that $H_*(V, d)$ is free and a free dga is not necessary quasi perfect. We extend Baues' theorem as follows:

Two perfect (or quasi-perfect) dgas are quasi-isomorphic if and only if their Whitehead exact sequences are coherently isomorphic.

As a consequence we establish the following topological result:

Let X and Y be two simply connected CW*-complexes such that $H_*(X, R)$ and $H_*(Y, R)$ are free and let $AH(X)$ and $AH(Y)$ be their respective Adams–Hilton models* [1]. *Then $AH(X)$ and $AH(Y)$ are quasi-isomorphic if and only if their Whitehead exact sequences are coherently isomorphic.*

For the third question we introduce the notion of a strong isomorphism between the Whitehead exact sequences associated with two given free dgas and we show that:

If the Whitehead exact sequences associated with two free dgas are strongly isomorphic, then the two given free dgas are quasi-isomorphic.

Consequently we derive:

Let X and Y be two simply connected CW*-complexes for which the Whitehead exact sequences associated with $AH(X)$ and $AH(Y)$ are strongly isomorphic, then $AH(X)$ and $AH(Y)$ are quasi-isomorphic.*

This article is organized as follows. In Section 2, the Whitehead exact sequences associated with free dgas are defined as well as perfect (quasi-perfect) dgas and their essential properties are given. Section 3 is devoted to the notion of coherent homomorphism as well as the notion of adapted systems which constitutes the technical part on which this work is based and that allows us to derive the main first results in this paper. At the end of Section 3 we give an algorithm showing how the obtained results can be used in order to compute the set of the quasi-isomorphism classes of perfect dgas and in the end of this section we illustrate our results by giving some geometric applications. Finally Section 4 is devoted to the proof of the second main result in this paper.

2. Whitehead exact sequence

Let $(T(V), \partial)$ be a free dga, where $V = V_{\geq 1}$. For all $n \geq 2$, let $T(V_{\leq n})$ be the free sub-dga of $T(V)$ generated by the graded module $(V_i)_{i \leq n}$. Define the pair $(T(V_{\leq k}), T(V_{\leq k-1}))$ as the quotient:

$$(T(V_{\leq k}), T(V_{\leq k-1})) = \frac{T(V_{\leq k})}{T(V_{\leq k-1})}.$$