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Embedding theorems for HNN extensions of inverse semigroups

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Abstract

A variant of an HNN extension of an inverse semigroup introduced by Gilbert [N.D. Gilbert, HNN extensions of inverse semigroups and groupoids, J. Algebra 272 (2004) 27–45] is defined provided that associated subsemigroups are order ideals. We show this presentation still makes sense without the assumption on associated subsemigroups in the sense that it gives a semigroup deserving to be an HNN extension, and it is embedded into another variant using the automata theoretical technique based on combinatorial and geometrical properties of Schützenberger graphs.

MSC: 20M18; 20M05; 68Q70

1. Introduction

The concept of an HNN extension of a group can be generalized to the class of semigroups in several fashions. One such generalization is studied in [1,2,6,8,9] where a stable letter belongs to the group of units. It can also be generalized to the class of inverse semigroups in a way that stable letters are not necessarily a group element. There are two such approaches by Gilbert [4] and Yamamura [15]. The first is constructed by interpreting the HNN extensions of groupoids considered by Higgins [7] under the assumption that associated subsemigroups are order ideals, and each stable letter corresponds to one of the idempotents in an associated subsemigroup. This approach has a strong connection with groupoid theory. The second has the features of a free construction in inverse semigroups. It is constructed under the assumption that associated subsemigroups are monoids, and only one stable letter is required.

In this paper, we clarify the relationship between these two variants of HNN extensions of inverse semigroups. First, we introduce several other variants of HNN extensions and generalize an HNN extension in the sense of [4] to a more general context so that the embeddability can still hold even though associated subsemigroups are not order ideals. Second, we show that every HNN extension in the sense of [4] and its generalization can be naturally embedded into another variant of HNN extensions introduced in [14]. This implies that the HNN extensions in the sense of [4] are actually subsemigroups of the other variant of HNN extensions in [14,15]. Third, we give a necessary and sufficient condition for the semilattice of idempotents of an HNN extension to coincide with that of the original inverse semigroup.

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Our main tool in this paper is the automata theoretic method using Schützenberger graphs introduced by Stephen [13]. To obtain the second result above, we use the iterative production of approximate automata of Schützenberger graphs. To be precise, we establish a simulation among approximate automata. The reader is referred to [13] for Schützenberger graphs and [12] for standard terminology in semigroup theory.

2. Concepts of HNN extensions of inverse semigroups

We recall several concepts of an HNN extension of an inverse semigroup in [4,14,15], and also introduce some presentations as a candidate for an HNN extension. Then we examine their properties. In the rest of the paper, we suppose that S is an inverse semigroup, A and B are isomorphic inverse subsemigroups of S, and ϕ is an isomorphism of A onto B.

2.1. Presentation $S(\phi, t)$

We now suppose $e_A \in A \subset e_A S e_A$ and $e_B \in B \subset e_B S e_B$, where e_A and e_B are idempotents. The inverse semigroup $S(\phi, t)$ is defined by the presentation

$$\mathrm{Inv}(S, t \mid t^{-1}at = \phi(a) \text{ for } \forall a \in A, t^{-1}t = e_B, tt^{-1} = e_A). \tag{2.1}$$

The element t in $S(\phi, t)$ is called the *stable letter*. The most important property of this construction is that S is naturally embedded into $S(\phi, t)$. This is classified as an *HNN extension of type I* in [14] and the restricted case that the stable letter belongs to the group of units is discussed in [1,2,6,8,9]. This construction is applied to several algorithmic problems like the undecidability of Markov properties of inverse semigroups in [15,18]. There are many concrete examples that admit a natural decomposition as an HNN extension (2.1). For example, free groups, free inverse semigroups, the bicyclic monoid, free Clifford semigroups and Bruck–Reilly extensions admit a natural HNN extension decomposition (see [15,18]). It is also clear that an HNN extension of a group is an HNN extension in the sense of (2.1). We also remark that lower bounded HNN extensions are discussed in [10] and an inverse semigroup whose defining relations have the form $d_1 = d_2$, where d_1 and d_2 are Dyck words, admits a decomposition as an HNN extension of a semilattice [16].

An HNN extension (2.1) is called *full* if $E(A) = E(B) = E(S)$. A full HNN extension can be characterized as a fundamental inverse monoid of a loop of inverse monoids, and this is employed to study the class of inverse monoids acting on ordered forests in [19]. This is considered as a generalization of the Bass–Serre theory. A normal form is given for locally full HNN extensions in [18]. Recall that an inverse submonoid A ($e \in A \subset eSe$) is called *locally full* if $E(A) = E(eSe)$ and an HNN extension (2.1) is called *locally full* if A and B are locally full [17].

2.2. Presentation $S[\phi, t_e]$

The inverse semigroup $S[\phi, t_e]$ is defined by the presentation

$$\mathrm{Inv}(S, t_e (e \in E(A)) \mid t_{aa^{-1}}^{-1} a t_{a^{-1}a} = \phi(a) \text{ for } \forall a \in A, t_e^{-1} t_f = \phi(e)\phi(f), t_e t_f^{-1} = ef). \tag{2.2}$$

We here denote the set of idempotents of A by $E(A)$. The elements t_e ($e \in E(A)$) in $S[\phi, t_e]$ are called the *stable letters*. Interpreting the concept of an HNN extension of a groupoid given by Higgins [7], Gilbert [4] studies the inverse semigroups presented by (2.2) provided that A and B are order ideals of S, and denotes it by $S_{*A,\phi}$. As a matter of fact, he adopts the relation $t_e^{-1} t_f = \phi(ef)$ instead of $t_e^{-1} t_f = \phi(e)\phi(f)$ in (2.2). These two relations are equivalent and one can choose either of them. However, we adopt $t_e^{-1} t_f = \phi(e)\phi(f)$ because for technical reasons as we will see later. Every element in (2.2) has a certain normal form and S is naturally embedded into $S[\phi, t_e]$ provided that A and B are order ideals [4]. Gilbert's perspective is groupoid theoretic, and the assumption that A and B are order ideals is critical to obtain a relatively easy groupoid structure. We remark that finite presentations of such a presentation is discussed in [3].

We shall show that the construction (2.2) still makes sense even though A and B are not order ideals in the sense that the natural mapping $s \mapsto s$ ($s \in S$) is an embedding of S into $S[\phi, t_e]$. Unless A and B are order ideals, the inverse semigroup presented by (2.2) has more complicated groupoid structure than S. In fact, we shall show that in Theorem 6.4 the set of vertices of the corresponding groupoid for $S[\phi, t_e]$ is equal to that of S if and only if A and B are order ideals.