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# A uniform Artin–Rees property for syzygies in rings of dimension one and two

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**Abstract**

Let $(R, \mathbf{m}, \mathsf{k})$ be a local Noetherian ring, let $M$ be a finitely generated $R$-module and let $I \subset R$ be an $\mathbf{m}$-primary ideal. Let $\mathbf{F} = \{F_i, \partial_i\}$ be a free resolution of $M$. In this paper we study the question whether there exists an integer $h$ such that $I^n F_i \cap \ker(\partial_i) \subset I^{n-h} \ker(\partial_i)$ holds for all $i$. We give a positive answer for rings of dimension at most two. We relate this property to the existence of an integer $s$ such that $I^s$ annihilates the modules $\mathrm{Tor}_i^R(M, R/I^n)$ for all $i > 0$ and all integers $n$.


*MSC:* 13C10

## 1. Introduction

In this paper $(R, \mathbf{m}, \mathsf{k})$ denotes a local Noetherian ring, and all modules are finitely generated. As general reference we refer to [1,4].

Let $I$ be an ideal of $R$, let $M$ be an $R$-module and $N$ a submodule of $M$. The Artin–Rees lemma states that there exists an integer $h$ depending on $I$, $M$ and $N$ such that for all $n \geq h$ one has

$$I^n M \cap N = I^{n-h}(I^h M \cap N). \tag{1.0.1}$$

A weaker property, which is often the one used in applications, is

$$I^n M \cap N \subset I^{n-h} N. \tag{1.0.2}$$

Much work has been done to determine whether $h$ can be chosen uniformly, in the sense that (1.0.2) would be satisfied simultaneously for every ideal belonging to a given family; see [3,6,8–11]. We study another kind of uniformity.

**Theorem 1.1.** *Let $(R, \mathbf{m}, \mathsf{k})$ be a local Noetherian ring with $\dim R \leq 2$. Let $M$ a finitely generated $R$-module and $I \subset R$ an $\mathbf{m}$-primary ideal. There exists an integer $h$ such that for every free resolution $\mathbf{F} = \{F_i, \partial_i^{\mathbf{F}}\}$ of $M$ there are inclusions*

$$I^n \mathbf{F}_{i-1} \cap \ker(\partial_i^{\mathbf{F}}) \subseteq I^{n-h} \ker(\partial_i^{\mathbf{F}}) \quad \textit{for all } i \geq 1 \textit{ and all } n > h. \tag{1.1.1}$$

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The main motivation for this work is a theorem due to Eisenbud and Huneke [5, Theorem 3.1]: Let $M$ be an $R$-module and let $\mathbf{F} = \{F_i, \partial_i^{\mathbf{F}}\}$ be a free resolution of $M$. If for every non-maximal prime ideal $\mathbf{p}$ of $R$ the $R_{\mathbf{p}}$-module $M_{\mathbf{p}}$ has finite projective dimension and its rank is independent of $\mathbf{p}$, then there exists an integer $h$ such that (1.1.1) holds.

To prove Theorem 1.1 we study the annihilators of the modules $\mathrm{Tor}_i^R(M, R/I^n)$; see also [5, Proposition 4.1].

**Theorem 1.2.** *Let $(R, \mathbf{m}, \mathsf{k})$ be a local Noetherian ring, let $r$ be an integer and let $\mathcal{F}$ be a family of ideals. Assume that one of the following conditions holds:*

(1) $\dim R = 1$, $r = 2$ *and $\mathcal{F}$ is the family of all $\mathbf{m}$-primary ideals;*
(2) $\dim R = 2$, $r = 3$ *and $\mathcal{F}$ is the family of all parameter ideals.*

*Then there exists an integer $h$ such that*

$$I^h \mathrm{Tor}_j^R(M, R/I^n) = 0$$

*for every $R$-module $M$, every integer $n$, every $j \geq r$ and every $I \in \mathcal{F}$.*

In the next section we define syzygetically Artin–Rees modules and study the case where the ring is Cohen–Macaulay. In Section 3 we study uniform annihilators for certain Tor-modules. In Section 4 we prove Theorems 1.2 and 1.1 (see Theorems 4.4 and 4.5) for rings of dimension one, and in Section 5 we prove them (see Theorems 5.4 and 6.1) for rings of dimension two.

## 2. Syzygetically Artin–Rees modules

Given an $R$-module $M$ and $\mathbf{F} = \{F_i, \partial_i^{\mathbf{F}}\}$ a minimal free resolution of $M$, we define $\Omega_i^R(M) := \ker(\partial_{i-1}^{\mathbf{F}})$.

**Lemma 2.1.** *Let $M$ be an $R$-module and let $I$ be an ideal of $R$. Let $h$ be an integer. The following conditions are equivalent:*

(1) *for every free resolution $\mathbf{G} = \{G_i, \partial_i^{\mathbf{G}}\}$ one has:*

$$I^n G_i \cap \ker(\partial_i^{\mathbf{G}}) \subset I^{n-h} \ker(\partial_i^{\mathbf{G}}) \quad \text{for all } i \geq 1 \text{ and all } n > h; \tag{2.1.1}$$

(2) *for some free resolution $\mathbf{G} = \{G_i, \partial_i^{\mathbf{G}}\}$ inclusion (2.1.1) holds.*

**Proof.** For every free resolution $\mathbf{G} = \{G_i, \partial_i^{\mathbf{G}}\}$, we can write $G_i = F_i \oplus C_i \oplus D_i$, where $\partial_i^{\mathbf{G}}|_{F_i} \subseteq \mathbf{m}F_{i-1}$, $\partial_i^{\mathbf{G}}(D_i) = 0$ and $\partial_i^{\mathbf{G}}(C_i) = C_{i-1}$. In particular, the inclusion $I^n G_i \cap \ker(\partial_i^{\mathbf{G}}) I^{n-h} \ker(\partial_i^{\mathbf{G}})$ holds for all $i > 0$ and $n > h$ for a free resolution $\mathbf{G}$ of $M$ if and only if it holds for the minimal free resolution $\mathbf{F}$ of $M$. $\square$

**Definition 2.2.** Let $(R, \mathbf{m}, \mathsf{k})$ be a local Noetherian ring. Let $M$ be a finitely generated $R$-module, let $I$ be an ideal of $R$ and let $h$ be an integer. An $R$-module $M$ is *syzygetically Artin–Rees* of level $h$ with respect to $I$ if one of the equivalent conditions of Lemma 2.1 holds.

Let $\mathcal{F}$ be a family of ideals. If there exists an integer $h$ such that (2.1.1) holds for every ideal $I \in \mathcal{F}$ then we say that $M$ is *syzygetically Artin–Rees* with respect to $\mathcal{F}$, or simply *syzygetically Artin–Rees* if $\mathcal{F}$ is the family of all ideals.

### 2.3. Uniform Artin–Rees

Let $(R, \mathbf{m}, \mathsf{k})$ be a local Noetherian ring. Given an $R$-module $M$ and a submodule $N$, there exists an integer $h = h(M, N)$ such that $I^n M \cap N \subset I^{n-h} N$, for every ideal $I$ of $R$ and every $n > h$. See [6, Theorem 4.12].

**Lemma 2.4.** *Let $M$ be an $R$-module and let $\mathcal{F}$ be a family of ideals. Then the following hold:*

(1) *$M$ is syzygetically Artin–Rees with respect to $\mathcal{F}$ if and only if $\Omega_i^R(M)$ is syzygetically Artin–Rees with respect to $\mathcal{F}$ for some integer $i > 0$.*
(2) *Let $R \to S$ be a faithfully flat extension. If $M \otimes_R S$ is syzygetically Artin–Rees with respect to the family of ideals $IS$ where $I \in \mathcal{F}$, then $M$ is syzygetically Artin–Rees with respect to $\mathcal{F}$.*