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Copositive matrices with circulant zero support set

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ABSTRACT

Let $n \geq 5$ and let $u^1, \dots, u^n$ be nonnegative real n-vectors such that the indices of their positive elements form the sets $\{1, 2, \dots, n-2\}, \{2, 3, \dots, n-1\}, \dots, \{n, 1, \dots, n-3\}$, respectively. Here each index set is obtained from the previous one by a circular shift. The set of copositive forms which vanish on the vectors $u^1, \dots, u^n$ is a face of the copositive cone $\mathcal{C}^n$. We give an explicit semi-definite description of this face and of its subface consisting of positive semi-definite forms, and study their properties. If the vectors $u^1, \dots, u^n$ and their positive multiples exhaust the zero set of an exceptional copositive form belonging to this face, then we say it has minimal circulant zero support set, and otherwise non-minimal circulant zero support set. We show that forms with non-minimal circulant zero support set are always extremal, and forms with minimal circulant zero support sets can be extremal only if n is odd. We construct explicit examples of extremal forms with non-minimal circulant zero support set for any order $n \geq 5$, and examples of extremal forms with minimal circulant zero support set for any odd order $n \geq 5$. The set of all forms with non-minimal circulant zero support set, i.e., defined by different collections $u^1, \dots, u^n$ of zeros, is a submanifold of codimension $2n$, the set of all forms with minimal circulant zero support set a submanifold of codimension n.

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1. Introduction

Let $\mathcal{S}^n$ be the vector space of real symmetric $n \times n$ matrices. In this space, we may define the cone $\mathcal{N}^n$ of element-wise nonnegative matrices, the cone $\mathcal{S}_+^n$ of positive semi-definite matrices, and the cone $\mathcal{C}^n$ of copositive matrices, i.e., matrices $A \in \mathcal{S}^n$ such that $x^T Ax \geq 0$ for all $x \in \mathbb{R}_+^n$. Obviously we have $\mathcal{S}_+^n + \mathcal{N}^n \subset \mathcal{C}^n$, but the converse inclusion holds only for $n \leq 4$ [6, Theorem 2]. Copositive matrices which are not elements of the sum $\mathcal{S}_+^n + \mathcal{N}^n$ are called *exceptional*. Copositive matrices play an important role in non-convex and combinatorial optimization, see, e.g., [5] or the surveys [10,17,4,8]. Of particular interest are the exceptional extreme rays of $\mathcal{C}^n$.

A fruitful concept in the study of copositive matrices is that of zeros and their supports, initiated in the works of Baumert [2,3], see also [16,7], and [19] for further developments and applications. A non-zero vector $u \in \mathbb{R}_+^n$ is called a *zero* of a copositive matrix $A \in \mathcal{C}^n$ if $u^T Au = 0$. The *support* $\operatorname{supp} u$ of a zero u is the index set of its positive elements.

Note that each of the cones $\mathcal{N}^n$, $\mathcal{S}_+^n$, $\mathcal{C}^n$ is invariant with respect to a simultaneous permutation of the row and column indices, and with respect to a simultaneous pre- and post-multiplication with a positive definite diagonal matrix. These operations generate a group of linear transformations of $\mathcal{S}^n$, which we shall call $\mathcal{G}_n$.

Exceptional copositive matrices first appear at order $n = 5$. The *Horn matrix*

$$H = \begin{pmatrix} 1 & -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 \end{pmatrix}, \tag{1}$$

named after its discoverer Alfred Horn, and the other matrices in its $\mathcal{G}_5$-orbit have been the first examples of exceptional extremal copositive matrices [14]. Any other exceptional extremal matrix in $\mathcal{C}^5$ lies in the $\mathcal{G}_5$-orbit of a matrix

$$T(\theta) = \begin{pmatrix} 1 & -\cos\theta_1 & \cos(\theta_1+\theta_2) & \cos(\theta_4+\theta_5) & -\cos\theta_5 \\ -\cos\theta_1 & 1 & -\cos\theta_2 & \cos(\theta_2+\theta_3) & \cos(\theta_5+\theta_1) \\ \cos(\theta_1+\theta_2) & -\cos\theta_2 & 1 & -\cos\theta_3 & \cos(\theta_3+\theta_4) \\ \cos(\theta_4+\theta_5) & \cos(\theta_2+\theta_3) & -\cos\theta_3 & 1 & -\cos\theta_4 \\ -\cos\theta_5 & \cos(\theta_5+\theta_1) & \cos(\theta_3+\theta_4) & -\cos\theta_4 & 1 \end{pmatrix} \tag{2}$$

for some angles $\theta_k \in (0, \pi)$ satisfying $\sum_{k=1}^5 \theta_k < \pi$ [15, Theorem 3.1]. Both the Horn matrix H and the matrices $T(\theta)$, which are referred to as Hildebrand matrices, possess zeros with supports $\{1,2,3\}, \{2,3,4\}, \{3,4,5\}, \{4,5,1\}, \{5,1,2\}$, respectively. Note that these supports are exactly the vertex subsets obtained by removing the vertices of a single edge in the cycle graph C_5.