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New classes of matrix decompositions

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A B S T R A C T

The idea of decomposing a matrix into a product of structured matrices such as triangular, orthogonal, diagonal matrices is a milestone of numerical computations. In this paper, we describe six new classes of matrix decompositions over complex number field, extending our work in [5]. We prove that every $n \times n$ complex matrix is a product of finitely many tridiagonal, skew symmetric (when n is even), companion and generalized Vandermonde matrices, respectively. We also prove that a generic complex $n \times n$ centrosymmetric matrix is a product of finitely many symmetric Toeplitz (resp. persymmetric Hankel) matrices. We determine an upper bound of the number of structured matrices needed to decompose a matrix for each case.

1. Introduction

Matrix decomposition is an important technique in numerical computations. For example, we have classical matrix decompositions:

1. LU: a generic matrix can be decomposed as a product of an upper triangular matrix and a lower triangular matrix.

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2. QR: every real matrix can be decomposed as a product of an orthogonal matrix and an upper triangular matrix.
3. SVD: every matrix can be decomposed as a product of two unitary matrices and a diagonal matrix.

Those classical matrix decompositions (LU, QR and SVD decompositions) correspond to the Bruhat, Iwasawa, and Cartan decompositions of Lie groups [3,1]. Matrix decompositions different from those classical ones are, for instance:

1. Every $n \times n$ matrix is a product of $(2n + 5)$ Toeplitz (resp. Hankel) matrices [5].
2. Every matrix is a product of two symmetric matrices [2].

As we have seen for classical matrix decompositions, Toeplitz, Hankel and symmetric matrix decompositions are important in the sense that structured matrices are well understood. For example, a Toeplitz linear system can be solved in $O(n \log n)$ floating point operations using displacement rank [7], compared to at least $O(n^2)$ floating point operations for general linear systems. Sometimes the matrix decomposition refers to the decomposition of a matrix into the sum of two matrices, see for example, [12–14]. However, whenever we mention the matrix decomposition in this paper, we always refer to the multiplicative version.

In this article, we study matrix decompositions over $\mathbb{C}$ beyond those mentioned above. We use Algebraic Geometry as our tool to explore the existence of matrix decompositions for various structured matrices. We define necessary notions in Section 2 and we prove some general results for the matrix decomposition problem and establish a strategy to tackle the matrix decomposition problem in Section 3. In Section 4 we discuss the matrix decomposition problem with two factors and recover the LU decomposition and the QR decomposition for generic matrices using our method. Here the LU (resp. QR) decomposition for generic matrices means that the set of complex matrices which can be written as the product of a complex lower triangular (resp. complex orthogonal) matrix and a complex upper triangular matrix is a dense open subset (with the Zariski topology) of the space of all $n \times n$ matrices. Traditionally, the QR decomposition over $\mathbb{C}$ means the decomposition of a matrix into the product of an upper triangular matrix and a unitary matrix. However, since our method is based on Algebraic Geometry and the group of unitary matrices is not algebraic, we are not able to recover the traditional version of the QR decomposition over $\mathbb{C}$. Hence the Q in our decomposition is a complex orthogonal matrix, that is, a matrix in $\mathbb{C}^{n\times n}$ satisfying $Q^{\mathsf{T}} Q = \mathrm{Id}_n$, where Id_n is the $n \times n$ identity matrix.

In Section 5 we apply the strategy built in Section 3.4 to the matrix decomposition problem for linear subspaces. Lastly, in Section 6 we apply the strategy to the matrix decomposition problem for non-linear varieties. We summarize our contributions in the following list: