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A new kurtosis matrix, with statistical applications

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ABSTRACT

The number of fourth-order moments which can be obtained from a random vector rapidly increases with the vector's dimension. Scalar measures of multivariate kurtosis may not satisfactorily capture the fourth-order structure, and matrix measures of multivariate kurtosis are called for. In this paper, we propose a kurtosis matrix derived from the dominant eigenpair of the fourth standardized moment. We show that it is the best symmetric, positive semidefinite Kronecker square root approximation to the fourth standardized moment. Additional properties are derived for realizations from GARCH and reversible random processes. Statistical applications include independent component analysis and projection pursuit. The star product of matrices highlights the connection between the proposed kurtosis matrix and other kurtosis matrices which appeared in the statistical literature. A simulation study assesses the practical relevance of theoretical results in the paper.

1. Introduction

Let $x = (X_1, \ldots, X_d)^T$ be a real, d-dimensional random vector satisfying $E\left(X_i^4\right) < +\infty$, for $i = 1, \ldots, d$. The fourth moment matrix (henceforth fourth moment,

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for short) of x is the $d^2 \times d^2$ matrix $M_{4,x} = E\left(x \otimes x^T \otimes x \otimes x^T\right)$, where "$\otimes$" denotes the Kronecker product. It is also the second moment of the vector $x \otimes x$, and conveniently arranges all the fourth-order moments $\mu_{ijhk} = E\left(X_i X_j X_h X_k\right)$ of x, for $i, j, h, k = 1, \ldots, d$. In particular, it admits the block matrix representation $M_{4,x} = \{B_{pq}\}$, where $B_{pq} = E\left(X_p X_q x x^T\right)$, for $p, q = 1, \ldots, d$. If the variance Σ of x is positive definite, its fourth standardized moment $M_{4,z}$ is the fourth moment of $z = \Sigma^{-1/2}(x - \mu)$, where μ is the expectation of x and $\Sigma^{-1/2}$ is the symmetric, positive definite square root of the concentration matrix Σ^{-1}.

Statistical applications of the fourth moment include the covariance between quadratic forms of random vectors [8], asymptotic distribution of the sample covariance matrix (see, for example [15, page 285]), testing for elliptical symmetry [33], robust tests for covariance matrices [34], independent component analysis [12], models for multivariate financial data [10]. A variant of the fourth moment, which arranges in a different way all fourth-order centered moments of a random vector, appears in portfolio theory [13].

The number of possibly distinct elements in the fourth moment rapidly increases with the vector's dimension, thus impairing the interpretation of the fourth moment itself. For example, the fourth moments of 3-dimensional and 6-dimensional random vectors may contain 81 and 126 distinct elements. A natural solution would be summarize the fourth standardized moment with a scalar function of it, as done by Mardia [24], Malkovich and Afifi [23] and Koziol [17]. Unfortunately, scalar functions are more appropriate for hypothesis testing than for understanding multivariate kurtosis. Kollo [16] illustrates this limitation with two bivariate distributions with very different shapes but with the same value of Mardia's kurtosis.

Matrix-valued functions of the fourth standardized moment would be a reasonable compromise between detail and synthesis. Cardoso [5] and Mòri [26] independently proposed $K = E\left(z^T z z z^T\right)$ as a kurtosis matrix. Its statistical applications include independent component analysis [5], invariant co-ordinate selection [35] and cluster analysis [30]. It depends on $M_{4,z}$ only through $E\left(Z_i^2 Z_j^2\right)$, for $i, j = 1, \ldots, d$. In order to take into account all fourth-order moments, Kollo [16] proposed the kurtosis matrix $E\left(z^T 1_d 1_d^T z z z^T\right)$, where 1_d is the d-dimensional vector of ones. Both kurtosis matrices greatly reduce the number of kurtosis parameters. For example, the fourth moment and the kurtosis matrix of a 6-dimensional random vector might have up to 126 and 21 distinct parameters, respectively.

In this paper, we define a new kurtosis matrix in a different way. Rather than relying on intuitive arguments, or motivating it with a specific application, we look for the symmetric, positive semidefinite $d \times d$ matrix which best approximates the fourth standardized moment, in the Kronecker square root sense [27]. We shall refer to this matrix as to eigenkurtosis, as a reminder of its connection with the dominant eigenpair of the fourth standardized moment. Other properties follow under additional assumptions, as for example reversibility. Statistical applications include kurtosis-based projection pur-