Linear Algebra and its Applications 512 (2017) 48–70

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

The maximum of the minimal multiplicity of eigenvalues of symmetric matrices whose pattern is constrained by a graph ☆

Polona Oblak [a,*], Helena Šmigoc [b]

[a] *Faculty of Computer and Information Science, University of Ljubljana, Večna pot 113, SI-1000 Ljubljana, Slovenia*
[b] *School of Mathematical Sciences, University College Dublin, Belfield, Dublin 4, Ireland*

ARTICLE INFO

Article history:
Received 18 April 2016
Accepted 10 September 2016
Available online 22 September 2016
Submitted by S. Fallat

MSC:
05C50
15A18
15B57

Keywords:
Symmetric matrix
Multiplicity of an eigenvalue
Minimal rank
Graph

ABSTRACT

In this paper we introduce a parameter $\mathrm{Mm}(G)$, defined as the maximum over the minimal multiplicities of eigenvalues among all symmetric matrices corresponding to a graph G. We develop basic properties of $\mathrm{Mm}(G)$ and compute it for several families of graphs.

☆ This work was supported by Science Foundation Ireland under Grant 11/RFP.1/MTH/3157.
* Corresponding author.
E-mail addresses: polona.oblak@fri.uni-lj.si (P. Oblak), helena.smigoc@ucd.ie (H. Šmigoc).

http://dx.doi.org/10.1016/j.laa.2016.09.014

1. Introduction

Given a simple undirected graph $G = (V(G), E(G))$ with vertex set $V(G) = \{1, 2, \dots, n\}$, let $S(G)$ be the set of all real symmetric $n \times n$ matrices $A = (a_{ij})$ such that, for $i \neq j$, $a_{ij} \neq 0$ if and only if $(i, j) \in E(G)$. There is no restriction on the diagonal entries of A.

The question of characterizing all lists of real numbers

$$\{\lambda_1, \lambda_2, \dots, \lambda_n\}$$

that can be the spectrum of a matrix $A \in S(G)$, is known as the *Inverse Eigenvalue Problem for G*. This question and the related question of characterizing all possible multiplicities of eigenvalues of matrices in $S(G)$ have been studied primarily for trees [6,11,13,14]. A subproblem to the Inverse Eigenvalue Problem for graphs that has attracted a lot of attention over the years is that of minimizing the rank of all $A \in S(G)$. Finding the minimal rank of G, defined as

$$\mathrm{mr}(G) = \min\{\mathrm{rk}(A); A \in S(G)\},$$

is equivalent to finding the maximal multiplicity of an eigenvalue of $A \in S(G)$, denoted by $M(G)$. *The minimum rank problem* has been resolved for several families of graphs. We refer the reader to an excellent survey paper on the problem [8] where additional references can be found. A more recent survey paper [7] not only gives an up-to-date on the minimum rank problem, but it also talks about several of its variants that can be found in the literature. For example, the possible inertia of matrices $A \in S(G)$ has been studied in [2–4] and the minimum number of distinct eigenvalues in [1]. For a matrix A, we let $q(A)$ denote the number of distinct eigenvalues of A. For a graph G, we define

$$q(G) = \min\{q(A);\ A \in S(G)\}.$$

In [15] we considered the problem of determining for which graphs G there exists a matrix in $S(G)$ whose characteristic polynomial is a square, i.e. the multiplicities of all its eigenvalues are even. This question is closely related to the question of determining for which graphs G there exists a matrix in $S(G)$ with all the multiplicities of eigenvalues at least 2. In this paper we bring this topic further by defining and studying a new parameter for a graph G denoted by $\mathrm{Mm}(G)$. For a matrix $A \in M_n(\mathbb{R})$ we denote $\mathrm{Mm}(A)$ to be the minimal eigenvalue multiplicity of A. Then $\mathrm{Mm}(G)$ is defined to be:

$$\mathrm{Mm}(G) = \max\{\mathrm{Mm}(A); A \in S(G)\}.$$

In order words, we define $\mathrm{Mm}(G)$ to be the maximum over the minimal multiplicities of eigenvalues among all $A \in S(G)$. Clearly, $\mathrm{Mm}(G) \leq \lfloor \frac{n}{2} \rfloor$ for all nonempty graphs on n vertices. If $\mathrm{Mm}(G) = 1$, then all matrices in $S(G)$ must have a simple eigenvalue.