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Edge perturbation on graphs with clusters: Adjacency, Laplacian and signless Laplacian eigenvalues

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ABSTRACT

Let G be a simple undirected graph of order n. A cluster in G of order c and degree s, is a pair of vertex subsets (C, S), where C is a set of cardinality $|C| = c \geq 2$ of pairwise co-neighbor vertices sharing the same set S of s neighbors. Assuming that the graph G has $k \geq 1$ clusters $(C_1, S_1), \ldots, (C_k, S_k)$, consider a family of k graphs $H_1, \ldots, H_k$ and the graph $G(H_1, \ldots, H_k)$ which is obtained from G after adding the edges of the graphs $H_1, \ldots, H_k$ whose vertex set of each H_j is identified with C_j, for $j = 1, \ldots, k$. The Laplacian eigenvalues of $G(H_1, \ldots, H_k)$ remain the same, independently of the graphs $H_1, \ldots, H_k$, with the exception of $|C_1| + \cdots + |C_k| - k$ of them. These new Laplacian eigenvalues are determined using a unified approach which can also be applied to the determination of a same number of adjacency and signless Laplacian eigenvalues when the graphs $H_1, \ldots, H_k$ are regular. The Faria's lower bound on the multiplicity of the Laplacian eigenvalue 1 of a graph with pendant vertices is generalized. Furthermore, the algebraic connectivity and the Laplacian index of $G(H_1, \ldots, H_k)$ remain the same, independently of the graphs $H_1, \ldots, H_k$.

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1. Introduction

The Laplacian and signless Laplacian spectra of a graph G with a cluster (C, S), where C is a set of cardinality $|C| \geq 2$ of pairwise co-neighbor vertices sharing the same set S of vertices, were studied in [2], as well as when those graphs are perturbed by adding edges between pairs of co-neighbor vertices. The particular case of graphs with pendant vertices perturbed by adding edges linking them, was considered in [4,9–11]. The effects on several spectral invariants were analyzed in [9] and [10]. In this paper, we consider the adjacency, Laplacian and signless Laplacian matrix of a graph in a unified way, when a graph G of order n has $k \geq 1$ clusters $(C_1, S_1), \ldots, (C_k, S_k)$. With this approach, several spectral results are deduced. Namely, identifying the vertex sets of a family of graphs $H_1, \ldots, H_k$ with the vertices of G such that $V(H_1) = C_1, \ldots, V(H_k) = C_k$ and denoting the obtained graph by $G(H_1, \ldots, H_k)$ we may conclude that the Laplacian index and the algebraic connectivity of both graphs, G and $G(H_1, \ldots, H_k)$, are the same. On the other hand, when the family of k graphs $H_1, \ldots, H_k$ varies, all the Laplacian eigenvalues remain the same with the exception of $|C_1| + \cdots + |C_k| - k$ of them. These new Laplacian eigenvalues are determined using a unified approach which can also be applied to the determination of $|C_1| + \cdots + |C_k| - k$ adjacency and signless Laplacian eigenvalues when the graphs $H_1, \ldots, H_k$ are regular. Furthermore, we generalize the Faria's lower bound on the multiplicity of the Laplacian eigenvalue 1 of graphs with pendant vertices and this generalization is straightforward extended to the signless Laplacian eigenvalue 1 and to the adjacency eigenvalue 0. It will be also proved that the algebraic connectivity and Laplacian index of $G(H_1, \ldots, H_k)$ remain the same, independently of the graphs $H_1, \ldots, H_k$.

The paper is organized as follows. The next section is devoted to introduce the notation and to recall the basic concepts. A general result on the spectral effects in graphs with a cluster by adding edges among pairwise co-neighbor vertices is obtained in Section 3. This result is generalized to graphs with $k \geq 1$ pairwise disjoint clusters in Section 4. In Section 5 a generalization of Faria's lower bound on the multiplicity of 1 as Laplacian eigenvalue is deduced and it is also proved that the Laplacian index and the algebraic connectivity remains the same when the family of graphs $H_1, \ldots, H_k$ varies in $G(H_1, \ldots, H_k)$, assuming tat G has k clusters.

2. Notation and basic definitions

Let $G = (V(G), E(G))$ be a simple undirected graph on n vertices with vertex set $V(G)$ and edge set $E(G)$. Let $D(G)$ be the diagonal matrix of order n whose $(i, i)-$entry is the degree d_i of the $i-th$ vertex of G and let $A\,(G)$ be the adjacency matrix of G. The matrices $L(G) = D(G) - A(G)$ and $L^+(G) = D\,(G) + A\,(G)$ are the Laplacian and signless Laplacian matrix of G, respectively. $L(G)$ and $L^+(G)$ are both positive semidefinite and $(0, \mathbf{1}_n)$ is an eigenpair of $L\,(G)$ where $\mathbf{1}_n$ is the all $1-$ vector of size n. Fiedler [6] proved that G is a connected graph if and only if the second smallest eigenvalue of $L(G)$ is