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Ruling out certain 5-spectra for the symmetric nonnegative inverse eigenvalue problem ☆

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ABSTRACT

A method is developed to show that certain spectra cannot be realized for the S-NIEP. It is applied in the 5-by-5 case to rule out many spectra that were previously unresolved. These are all in the case of 3 positive and 2 negative eigenvalues as all other cases are now resolved. For spectra of the sort we discuss, a diagram is given of the spectra that are excluded here, as well as those trivially realizable, those realizable because of the trace 0 case and those that may also be excluded because of the J–L–L conditions. A small region remains unresolved; it is a very small fraction of the area of those spectra we consider.

The n-by-n symmetric nonnegative inverse eigenvalue problem, S-NIEP, asks which collections of n real numbers, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, occur as the spectrum of an n-by-n

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symmetric nonnegative matrix, counting multiplicities. Already for $n = 5$ this has proven a very challenging problem. In [6] a detailed discussion is given of many parts of the case $n = 5$, for positive trace. In the trace 0 case ($\sum_{i=1}^{n} \lambda_i = 0$) such spectra have recently been characterized for $n = 5$ [9], but the general (trace ≥ 0) case remains open. It is convenient to categorize by sub-cases, based upon the number of positive eigenvalues. In case that number is 1, 4, or 5 resolution of the 5-by-5 S-NIEP is straightforward. In [1], the unresolved cases were narrowed to some with 2 positive eigenvalues and most nontrivial ones with 3 positive eigenvalues. Here, we first note that all cases with 2 positive eigenvalues may be resolved, and then (principally) give a new method to rule out many unresolved spectra with 3 positive eigenvalues. Some ruled out spectra are known to be realizable for the 5-by-5 R-NIEP (which only requests a nonnegative, not necessarily symmetric matrix realizing the given eigenvalues).

For both the R-NIEP and S-NIEP, $\sum_{i=1}^{n} \lambda_i \geq 0$ is clearly necessary and by the Perron–Frobenius theory $\lambda_1 \geq |\lambda_n|$, *i.e.* λ_1 is the spectral radius, is also necessary. In case $n = 4$, these two conditions alone are necessary and sufficient for both the S-NIEP and R-NIEP (this is straightforward and may be found in [5], among other places). In addition, if $\lambda_1 = \lambda_2$, a "tie" for spectral radius, the matrix must be reducible, and the spectrum must be partitionable into (at least) 2 nonnegative spectra in lower dimensions. Another necessary condition, J–L–L [3,5], is based upon traces of powers:

$$\left(\sum_{i=1}^{n} \lambda_i^k\right)^m \leq n^{m-1} \sum_{i=1}^{n} \lambda_i^{km}, \quad k, m = 1, 2, \ldots. \tag{1}$$

Whenever there is just one positive eigenvalue, it is known that the trace condition is sufficient, as well as necessary for the S-NIEP [12,2,4]. When there are just 2 positive eigenvalues, it has been observed [8] that "partitioned majorization" is sufficient for the S-NIEP, *i.e.* if the nonpositive eigenvalues $\lambda_3 \geq \lambda_4 \geq \cdots \geq \lambda_n$ may be partitioned into 2 subsets such that the larger sum of the absolute values in one set is no more than λ_1 and $\lambda_1 + \lambda_2$ is at least $|\lambda_3| + \cdots + |\lambda_n|$. For $n \leq 5$, this condition is also necessary [5,7,6]. Note that for $n > 5$ this is not true is shown by the spectrum $7, 5, -1/2, -7/2, -4, -4$, which is symmetrically realizable [4, Example 8 with $\delta = 1/2$] but is not partitionable.

When $n = 5$, this leaves the unresolved cases for the S-NIEP (because the cases of 4 or 5 nonnegative eigenvalues are straightforward):

$$\begin{aligned} \lambda_1 > \lambda_2 \geq \lambda_3 > 0 > \lambda_4 \geq \lambda_5 \\ \lambda_1 + \lambda_5 \geq 0 \\ \sum_{i=1}^{5} \lambda_i > 0 \\ \text{and} \quad \lambda_1 + \lambda_2 + \lambda_4 + \lambda_5 < 0. \end{aligned}$$