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Isometries of Minkowski geometries

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ABSTRACT

In this paper we review the known facts on isometries of Minkowski geometries and prove some new results on them. We give the normal forms of two special classes of operators and also characterize the isometry group of Minkowski 3-spaces in which the unit sphere does not contain an ellipse.

1. Introduction

The one hundred year old concept of "Minkowski space" is a nice topic of recent geometric research. Nevertheless, the phrase "Minkowski space" is applied for two different theories: the theory of normed linear spaces and the theory of linear spaces with indefinite metric. It is interesting (see [10–12]) that these essentially distinct theories have

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similar axiomatic foundations. The axiomatic build-up of the theory of linear spaces with indefinite metric comes from H. Minkowski [25] and the similar system of axioms of normed linear spaces was introduced by Lumer much later in [19]. The first concept widely used in physics is the mathematical structure of relativity theory and thus its importance is without doubt. On the other hand, the importance of the second theory is based on the fact that a large part of modern functional analysis works in so-called normed spaces which are more general ones than inner product (or Hilbert) spaces. This motivates the introduction of the so-called semi-inner product which is an important tool of the corresponding investigations. Of course, in both of these two theories a lot of problems can be formulated or can be solved in the language of geometry. Our theme of interest is the theory of finite-dimensional, separable and real semi-inner product spaces. Such a normed space with the branches of its geometric properties is called *Minkowski geometry*.

Our purpose is to review the possible characterizations of the distinct transformation groups of Minkowski geometry, take into consideration the analytic theory and also the synthetic geometric-algebraic investigations. Through the paper we prove some new statements. We mention Theorem 5 and Theorem 10 which introduce normal forms for the adjoint abelian operators and isometries of a Minkowski n-space. Theorem 12 describes the isometry group of a Minkowski 3-space with the property that its unit sphere does not contain an ellipse. This latter result generalizes a theorem of H.Martini, M. Spirova and K. Strambach proved for non-Euclidean Minkowski planes.

2. Operator theory of Minkowski geometry

A generalization of inner product and inner product spaces was raised by G. Lumer in [19].

Definition 1 *([19])*. The semi-inner product (s.i.p.) on a complex vector space V is a complex function $[x, y] : V \times V \longrightarrow \mathbb{C}$ with the following properties:

s1: $[x + y, z] = [x, z] + [y, z]$,
s2: $[\lambda x, y] = \lambda[x, y]$ for every $\lambda \in \mathbb{C}$,
s3: $[x, x] > 0$ when $x \neq 0$,
s4: $|[x, y]|^2 \leq [x, x][y, y]$.

A vector space V with a s.i.p. is an s.i.p. space.

G. Lumer proved that an s.i.p. space is a normed vector space with norm $\|x\| = \sqrt{[x, x]}$ and, on the other hand, that every normed vector space can be represented as an s.i.p. space. In [8] J. R. Giles showed that all normed vector spaces can be represented as s.i.p. spaces with homogeneous second variable. Giles also introduced the concept of *continuous s.i.p. space* as an s.i.p. space having the additional property: For any unit