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On maps preserving operators of local spectral radius zero

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ABSTRACT

Let $\mathcal{L}(X)$ be the algebra of all bounded linear operators on a complex Banach space X. We describe surjective linear maps ϕ on $\mathcal{L}(X)$ that satisfy

$$\mathrm{r}_{\phi(T)}(x) = 0 \Longrightarrow \mathrm{r}_T(x) = 0$$

for every $x \in X$ and $T \in \mathcal{L}(X)$. We also describe surjective linear maps ϕ on $\mathcal{L}(X)$ that satisfy

$$\mathrm{r}_T(x) = 0 \Longrightarrow \mathrm{r}_{\phi(T)}(x) = 0$$

for every $x \in X$ and $T \in \mathcal{L}(X)$. Furthermore, we characterize maps ϕ (not necessarily linear nor surjective) on $\mathcal{L}(X)$ which satisfy

$$\mathrm{r}_{\phi(T)-\phi(S)}(x) = 0 \text{ if and only if } \mathrm{r}_{T-S}(x) = 0$$

for every $x \in X$ and $T, S \in \mathcal{L}(X)$.

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1. Introduction

Let $\mathcal{L}(X)$ be the algebra of all bounded operators on a complex Banach space X. The local spectral radius of an operator $T \in \mathcal{L}(X)$ at a point $x \in X$ is defined by

$$\mathrm{r}_T(x) = \limsup_{n \to +\infty} \| T^n x \|^{\frac{1}{n}} .$$

Recall that the quasi-nilpotent part of an operator $T \in \mathcal{L}(X)$ is given by

$$\mathrm{H}_0(T) := \{x \in X : \limsup_{n \to +\infty} \| T^n x \|^{\frac{1}{n}} = 0\}.$$

The problem of describing linear or additive maps on $\mathcal{L}(X)$ preserving the local spectra has been initiated by A. Bourhim and T. Ransford in [5], and continued by several authors; see for instance [2–4,6–8] and the references therein.

In [8], C. Costara described surjective linear maps on $\mathcal{L}(X)$ which preserve operators of local spectral radius zero at points of X. He showed that if $\phi : \mathcal{L}(X) \to \mathcal{L}(X)$ is a linear and surjective map such that for every $x \in X$ and $T \in \mathcal{L}(X)$, we have

$$\mathrm{r}_{\phi(T)}(x) = 0 \text{ if and only if } \mathrm{r}_T(x) = 0,$$

then there exists a nonzero scalar $\mu \in \mathbb{C}$ such that $\phi(T) = \mu T$ for all $T \in \mathcal{L}(X)$.

This result has been extended by Bourhim and Mashreghi in [4] where it is shown that if ϕ is a surjective (not necessarily linear) map on $\mathcal{L}(X)$ that satisfies

$$\mathrm{r}_{\phi(T)-\phi(S)}(x) = 0 \text{ if and only if } \mathrm{r}_{T-S}(x) = 0,$$

for every $x \in X$ and $T, S \in \mathcal{L}(X)$, then there are a nonzero scalar $\mu \in \mathbb{C}$ and an operator $A \in \mathcal{L}(X)$ such that $\phi(T) = \mu T + A$ for all $T \in \mathcal{L}(X)$.

In this paper, we start by studying surjective linear maps ϕ on $\mathcal{L}(X)$ such that either

$$\mathrm{H}_0(\phi(T)) \subset \mathrm{H}_0(T)$$

for all $T \in \mathcal{L}(X)$, or

$$\mathrm{H}_0(T) \subset \mathrm{H}_0(\phi(T))$$

for all $T \in \mathcal{L}(X)$. This will give characterizations of surjective linear maps ϕ on $\mathcal{L}(X)$, that preserve operators of local spectral radius zero in one direction; i.e.

$$\mathrm{r}_{\phi(T)}(x) = 0 \Longrightarrow \mathrm{r}_T(x) = 0$$

for every $x \in X$ and $T \in \mathcal{L}(X)$, or