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The uniform normal form of a linear mapping

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ABSTRACT

This paper gives a normal form for a linear mapping of a finite dimensional vector space over a field of characteristic 0 into itself, which yields a better description of its structure than the classical companion matrix. Finding this normal form does not use any factorization of the characteristic polynomial of the linear mapping and requires only a finite number of operations in the field to compute.
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Let V be a finite dimensional vector space over a field k of characteristic 0. Let $A : V \to V$ be a linear mapping of V into itself with characteristic polynomial χ_A. The goal of this paper is to determine a normal form for A, which describes its structure better than the classical companion matrix. Finding this normal form does not require knowing a factorization of χ_A and uses only a finite number of operations in the field k to compute.

The main result of [2] gives an algorithm, involving no factorization of χ_A and only a finite number of operations in the field k, which yields the Jordan decomposition of A, namely, writes A as a sum of commuting semisimple and nilpotent S and N parts,

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respectively. For more details see [4]. In what follows we will assume that S and N are known.

1. Nilpotent normal form

In this section we describe the well known Jordan normal form for a nilpotent linear transformation N.

A linear transformation $N : V \to V$ is said to be *nilpotent of index* n if there is an integer $n \geq 1$ such that $N^{n-1} \neq 0$ but $N^n = 0$. Suppose that for some positive integer ≥ 1 there is a nonzero vector v, which lies in $\ker N^\ell \setminus \ker N^{\ell-1}$. The set $\{v, Nv, \ldots, N^{\ell-1}v\}$ is a *Jordan chain* of *length* ℓ with *generating vector* v. The space V^ℓ spanned by the vectors in a given Jordan chain of length ℓ is a N-*cyclic subspace* of V. Because $N^\ell v = 0$, the subspace V^ℓ is N-invariant. Since $\ker N|V^\ell = \mathrm{span}\{N^{\ell-1}v\}$, the mapping $N|V^\ell$ has exactly one eigenvector corresponding to the eigenvalue 0.

Fact 1.1. Vectors in a Jordan chain of length ℓ are linearly independent.

With respect to the *standard basis* $\{v, Nv, \ldots N^{\ell-2}v, N^{\ell-1}v\}$ of V^ℓ the matrix of $N|V^\ell$ is the $\ell \times \ell$ matrix

$$\begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

which is a *Jordan block* of size ℓ. The Jordan normal form theorem [1, pp. 270–274] states

Fact 1.2. V is a direct sum of N-cyclic subspaces.

A suitable reordering of the basis giving the Jordan normal form of N is a basis of V, realizes the Young diagram of N. The elements of the Young diagram are given by a dark dot $\bullet$ or an open dot $\circ$ in Fig. 1.1 and the arrows give the action of N on the basis vectors. The columns of the Young diagram of N are Jordan chains with generating vector given by an open dot. The black dots form a basis for the image $\mathrm{im}\, N$ of N. The open dots form a basis for a complementary subspace of $\mathrm{im}\, N$ in V. The dots on or above the jth row of the Young diagram form a basis for $\ker N^j$ and the black dots in the first row form a basis for $\ker N \cap \mathrm{im}\, N$. Let r_j be the number of dots in the jth row. Then $r_j = \dim \ker N^j - \dim \ker N^{j-1}$. Thus the Young diagram of N is unique.

We note that finding the generating vectors of the Young diagram of N or equivalently the Jordan normal form of N, involves solving linear equations with coefficients in the field k and thus requires only a finite number of operations in the field k to be determined.

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