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The quadrifocal variety

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A B S T R A C T

Multi-view Geometry is reviewed from an Algebraic Geometry perspective and multi-focal tensors are constructed as equivariant projections of the Grassmannian. A connection to the principal minor assignment problem is made by considering several flatlander cameras. The ideal of the quadrifocal variety is computed up to degree 8 (and partially in degree 9) using the representations of $\mathrm{GL}(3)^{\times 4}$ in the polynomial ring on the space of $3 \times 3 \times 3 \times 3$ tensors. Further representation-theoretic analysis gives a lower bound for the number of minimal generators.

1. Introduction and background

1.1. The multi-view variety

Multi-view Geometry is a branch of Computer Vision [22]. An important task in Computer Vision is to efficiently reconstruct the 3-dimensional scene from the 2-dimensional projections. Typically, one first estimates the multi-focal tensor associated to the n views using correspondences arising from one object seen in multiple images. From the

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multi-focal tensor one reconstructs the camera matrices. After the camera matrices are known, one uses the point correspondences to triangulate the 3D points.

In the standard pinhole camera model, the projection of 3D world points to multiple 2D images is represented by a collection of 3×4 matrices $(A_1, \ldots, A_n)$ and the mapping

$$\begin{array}{rcl} \mathbb{P}^3 & \to & \mathbb{P}^2 \times \cdots \times \mathbb{P}^2 \\ [x] & \mapsto & ([A_1 x], \ldots, [A_n x]). \end{array} \tag{1}$$

For cameras in general position the multi-view mapping (1) defines a 3 dimensional subvariety of the Cartesian product of projective spaces called the *multi-view* variety. Aholt, Sturmfels, and Thomas demonstrated rich algebraic geometry arising from this construction in [5]. They utilized a certain Hilbert scheme to describe the multi-view variety, its defining ideal, and further algebraic properties. Their theoretical techniques included Borel fixed monomial ideals, a universal Gröbner basis, degeneration to a special monomial ideal, and more. This work catalyzed a new area coined "Algebraic Vision" by Sameer Agarwal and Rekha Thomas. The reader may wish to consult the following other examples of recent work in this field [1–4,25].

1.2. Moduli spaces and quadrifocal tensors

Suppose the entries of the camera matrices are not known, but are considered as parameters. By modding out by the projective rescaling in each camera plane we have a moduli space of camera matrices. The algebraic varieties of multi-focal tensors are models for these moduli spaces, and we want to know their basic algebraic properties. Bifocal tensors are just 3×3 matrices of rank 2, defined by the 3×3 determinant. Chris Aholt and the author resolved the long-standing open question of describing the ideal of trifocal tensors [4], building on work of Alzati and Tortora [6]. They also computed its algebraic degree and Hilbert polynomial using Maple, Macaulay2 [17], and Bertini [7].

In this paper we will be mainly concerned with the *quadrifocal variety*. One may record the correspondences induced from one 3D point seen in 4 images by the 81 special 4×4 minors of stacked camera matrix $A = (A_1^\top | A_2^\top | \cdots | A_n^\top)$ that only use one column from each of the first four blocks of A. These coordinates are linear in each block, yielding a $3 \times 3 \times 3 \times 3$ *quadrifocal* tensor. The quadrifocal variety is the Zariski closure in $\mathbb{P}^{80}$ of the set of quadrifocal tensors.

We seek a complete description of the polynomial defining equations of the quadrifocal variety. The main result of the present article is a first step in this direction.

Theorem* 1.1. *Let I_d denote the degree d piece of the ideal of the quadrifocal variety.*

I_d is zero for $d < 3$.

I_3 is 600-dimensional.

I_4 is 48,600-dimensional but contains no minimal generators.

I_5 is 1,993,977-dimensional and contains at least 1,377 minimal generators.

I_6 is 54,890,407-dimensional and contains at least 37,586 minimal generators.