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On matrix polynomials with the same finite and infinite elementary divisors ☆

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ABSTRACT

A criterion is presented that characterizes when two matrix polynomials of any size, rank and degree have the same finite and infinite elementary divisors. This characterization inherits a coprimeness condition of the extended unimodular equivalence defined by Pugh and Shelton [17] in the set of real or complex matrix polynomials satisfying the constraint that the difference between the number of rows and columns is constant. This extended unimodular equivalence is first generalized to matrices of any size with elements in any principal ideal domain.

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1. Introduction

The problem of characterizing when two matrix polynomials (of any sizes, ranks and degrees) have the same finite and infinite elementary divisors has received attention in the literature of linear control systems (see [10,12–14] for example). Motivation and an account of relevant results about this problem can be found in the introduction of [14].

The problem originated in a paper by Pugh and Shelton, [17], which, in turns, was motivated by an equivalence relation defined by Fuhrmann [8] on polynomial system matrices (see [18,11]) that generalized the well-known Rosenbrock's strict system equivalence [18]. Pugh and Shelton called their equivalence relation *extended unimodular equivalence* and it is defined as follows: Let $\mathscr{P}(n_1, n_2)$ be the set of $(\ell + n_1) \times (\ell + n_2)$ matrix polynomials where n_1 and n_2 are fixed integers and ℓ is any integer greater than $\max(-n_1, -n_2)$. Then $D_1, D_2 \in \mathscr{P}(n_1, n_2)$ are said to be extended unimodular equivalent if there exist two matrix polynomials of appropriate sizes N_1 and N_2 such that:

(i) D_1 and N_1 are right coprime;
(ii) D_2 and N_2 are left coprime; and
(iii) $N_2 D_1 = D_2 N_1$.

An important feature of this equivalence relation is that it is defined for matrices with possible different sizes. These matrices cannot have the same Smith normal form (see (3)) but they may have the same invariant factors up to the number of them equal to 1. Since the elementary divisors of a matrix polynomial are determined by its non-trivial invariant factors (i.e., those different from 1), matrix polynomials of different sizes may have the same elementary divisors. This is precisely one of the main results in [17]: $D_1, D_2 \in \mathscr{P}(n_1, n_2)$ are extended unimodular equivalent if and only if they have the same Smith normal form up to the number of trivial invariant factors and the number of zero rows and zero columns.

Motivated by the study of the forward and backward behavior of linear homogeneous matrix difference equations (see [12–14] and the references therein), tries have been made to obtain an equivalence relation that preserves both the finite and infinite elementary divisors and generalizes the extended unimodular equivalence. In particular, a new relation is defined in [13], called *strong equivalence*, that, in addition to conditions (i)–(iii) above, requires similar properties for the reversal matrix polynomials of D_1 and D_2. It is then proved that strong equivalent matrix polynomials in $\mathscr{P}(n_1, n_2)$ have the same finite and infinite elementary divisors. However, it follows from [6, Theorem 4.1] that the converse is not true in general. Indeed, in that paper a new equivalence relation in $\mathscr{P}(n_1, n_2)$ is defined, called *spectral equivalence*, and it is shown that:

- $D_1, D_2 \in \mathscr{P}(n_1, n_2)$ are strong equivalent if and only if they are spectral equivalent.