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Necessary and sufficient conditions for solvability of the Riccati inequalities in the general case

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ABSTRACT

This paper presents necessary and sufficient conditions for solvability of Riccati inequalities in the general case in terms of purely imaginary eigenvalues of associated Hamiltonian matrices.

1. Introduction

The Riccati inequalities have been intensively studied for many decades [1–3]. The famous Kalman–Yakubovich–Popov (KYP) lemma [4,5] in its first variant also deals with solutions of inequalities which are equivalent to the Riccati inequalities arising in the theory of absolute stability [6–8]. In all these papers the quadratic terms of the Riccati inequalities are sign semidefinite. In this case they can be solved using efficient interior point algorithms developed for Linear Matrix Inequalities (LMI) [9].

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In the theory of H_∞ control there appear Riccati inequalities with sign indefinite quadratic terms. The frequency domain inequalities are no longer necessary for solvability of such inequalities, and the standard LMI-based approach is not applicable in such cases.

In this paper we present necessary and sufficient conditions for solvability of the Riccati inequalities in general form (without any constraints on the matrices involved). The approach is based on analysis of associated Hamiltonian matrices and Krein classification of eigenvalues of such matrices [10]. We use essentially the canonical representation of Hamiltonian matrices [11,12]. A contribution of this paper concerns also the ways how to move purely imaginary eigenvalues of Hamiltonian matrix R off the imaginary axis by adding terms ΔR such that $J\Delta R \geq 0$.

A similar problem is stated and solved in a fine paper [14], which finds the smallest (in an appropriate norm) Hamiltonian matrix ΔR such that, for the resulting matrix $R + \Delta R$, an arbitrarily small perturbation moves all the eigenvalues off the imaginary axis.

In our case we solve a similar problem, but the perturbations must satisfy a certain sign condition ($J\Delta R \geq 0$), the norm of ΔR is not restricted, and the question concerns the existence of such perturbations. It turns out that the Riccati inequality is solvable if and only if such a perturbation ΔR exists for an associated Hamiltonian matrix R.

The organization of the paper is the following. In Section 2 the problem is formulated, and the main notation is introduced, in particular, the associated matrices R and J. The case of Hamiltonian matrices without purely imaginary eigenvalues is considered in Section 3. In particular, it is shown that in this case the Riccati inequalities have solutions. In Section 4 it is shown that the Riccati inequality with a Hamiltonian matrix R_1 has a solution if another Riccati inequality with a Hamiltonian matrix R_2 has a solution and $JR_2 \geq JR_1$. Section 5 is devoted to Riccati inequalities with Hamiltonian matrices having only simple purely imaginary eigenvalues. A sufficient condition for solvability of such inequalities is derived. Section 6 presents crucial results allowing the elimination of Jordan blocks with purely imaginary eigenvalues and dimension greater than one by arbitrarily small perturbations of Hamiltonian matrices.

Important results which allow to destroy Jordan blocks of dimension bigger than one with purely imaginary eigenvalues using arbitrary small perturbations of the Hamiltonian matrices are presented in Section 6. Sections 7–9 are devoted to necessary and sufficient conditions of solvability of the Riccati inequalities. In all Sections we consider separately the case of Riccati inequalities with complex coefficients and the Riccati inequalities with real coefficients. An example in Section 10 illustrates the main results.

We use the following notation: I is the identity matrix; X^* is the conjugate transpose of matrix X; $^-$ is operation of complex conjugation; $span(X)$ is the minimal space which contains all columns of matrix X; $\|\cdot\|$ denotes the Euclidean norm; an eigenvalue λ_0 of a matrix M is called simple if the multiplicity of the root $\lambda = \lambda_0$ of the polynomial $\det(\lambda I - M)$ is equal to one.