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Lie derivations of incidence algebras

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ABSTRACT

Let X be a locally finite preordered set, $\mathcal{R}$ a commutative ring with identity and $I(X,\mathcal{R})$ the incidence algebra of X over $\mathcal{R}$. In this note we prove that each Lie derivation of $I(X,\mathcal{R})$ is proper, provided that $\mathcal{R}$ is 2-torsion free.

1. Introduction and preliminaries

Let A be an associative algebra over a commutative ring $\mathcal{R}$ (possibly without identity). We define the *Lie product* $[x,y] := xy - yx$ and *Jordan product* $x \circ y := xy + yx$ for all $x, y \in A$. Then $(A, [\ ,\ ])$ is a Lie algebra and $(A, \circ)$ is a Jordan algebra. It is fascinating to study the connection between the associative, Lie and Jordan structures on A. In this field, two classes of maps are of crucial importance. One of them consists of

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maps, preserving a type of product, for example, Lie homomorphisms etc. The other one is formed by differential operators, satisfying a type of Leibniz formulas, for example, Jordan derivations etc.

We recall that an $\mathcal{R}$-linear map $D : A \to A$ is called a *derivation* if $D(xy) = D(x)y + xD(y)$ for all $x, y \in A$, and it is called a *Lie derivation* if

$$D([x, y]) = [D(x), y] + [x, D(y)]$$

for all $x, y \in A$. Note that if D is a derivation of A and F is an $\mathcal{R}$-linear map from A into its center, then $D+F$ is a Lie derivation if and only if F annihilates all commutators $[x, y]$. A Lie derivation of the form $D + F$, with D being a derivation and F a central-valued map, will be called *proper*. In this article we find a class of algebras on which every Lie derivation is proper.

In the AMS Hour Talk of 1961 Herstein proposed many problems concerning the structure of Jordan and Lie maps in associative simple and prime rings [9]. Roughly speaking, he conjectured that these maps are all of the proper or standard form. The renowned Herstein's Lie-type mapping research program was formulated since then. Martindale gave a major force in this program under the assumption that the rings contain some nontrivial idempotents, see [15] for example. The first idempotent-free result on Lie-type maps was obtained by Brešar in [3]. We refer the reader to Brešar's survey paper [4] for a much more detailed historic background.

Let us now recall another notion, incidence algebra [13,19], with which we deal in this paper. Let $(X, \le)$ be a locally finite preordered set. This means that $\le$ is a reflexive and transitive binary relation on X, and for any $x \le y$ in X there are only finitely many elements z satisfying $x \le z \le y$. Given a commutative ring with identity $\mathcal{R}$, the *incidence algebra* $I(X, \mathcal{R})$ of X over $\mathcal{R}$ is defined to be the set

$$I(X, \mathcal{R}) := \{f : X \times X \to \mathcal{R} \mid f(x, y) = 0 \text{ if } x \not\le y\}$$

with algebraic operations given by

$$\begin{aligned}(f + g)(x, y) &= f(x, y) + g(x, y),\\ (rf)(x, y) &= rf(x, y),\\ (fg)(x, y) &= \sum_{x \le z \le y} f(x, z)g(z, y)\end{aligned}$$

for all $f, g \in I(X, \mathcal{R})$, $r \in \mathcal{R}$ and $x, y \in X$. The product fg is usually called the *convolution* in function theory. The identity element δ of $I(X, \mathcal{R})$ is given by $\delta(x, y) = \delta_{xy}$ for $x \le y$, where $\delta_{xy} \in \{0, 1\}$ is the Kronecker delta. It is clear that the full matrix algebra $\mathrm{M}_n(\mathcal{R})$ and the upper triangular matrix algebra $\mathrm{T}_n(\mathcal{R})$ are special examples of incidence algebras.

The incidence algebra of a partially ordered set (poset) X was first considered by Ward in [22] as a generalized algebra of arithmetic functions. Rota and Stanley devel-