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Restricted one-dimensional central extensions of restricted simple Lie algebras

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ABSTRACT

We study the restricted one-dimensional central extensions of an arbitrary finite dimensional restricted simple Lie algebra for $p \geq 5$. For $H^2(\mathfrak{g}) = 0$, we explicitly describe the cocycles spanning $H^2_*(\mathfrak{g})$, and in the case $H^2(\mathfrak{g}) \neq 0$, we give a procedure to describe a basis for $H^2_*(\mathfrak{g})$.

1. Introduction

In [3], the authors give an explicit description of the cocycles parameterizing the space of restricted one-dimensional central extensions of the Witt algebra $W(1) = W(1, \mathbf{1})$ defined over fields of characteristic $p \geq 5$. The Witt algebra is a finite dimensional

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restricted simple Lie algebra, and such algebras have been completely classified for primes $p \geq 5$ [10]. In this paper, we study the restricted one-dimensional central extensions of an arbitrary finite dimensional restricted simple Lie algebra for $p \geq 5$.

The one-dimensional central extensions of a Lie algebra $\mathfrak{g}$ defined over a field $\mathbb{F}$ are classified by the Lie algebra cohomology group $H^2(\mathfrak{g}) = H^2(\mathfrak{g}, \mathbb{F})$ where $\mathbb{F}$ is taken as a trivial $\mathfrak{g}$-module. The restricted one-dimensional central extensions of a restricted Lie algebra $\mathfrak{g}$ over $\mathbb{F}$ are likewise classified by the restricted Lie algebra cohomology group $H^2_*(\mathfrak{g}) = H^2_*(\mathfrak{g}, \mathbb{F})$. We refer the reader to [3] and [4] for descriptions of the complexes used to compute the ordinary and restricted Lie algebra cohomology groups as well as a review of the correspondence between one-dimensional central extensions (restricted central extensions) and the second cohomology (restricted cohomology) group. In particular, we adopt the notation and terminology in [3].

The dimensions of the ordinary cohomology groups $H^2(\mathfrak{g})$ for finite dimensional simple restricted Lie algebras are known [1,2,5,6,9]. Following the technique used in [12], we use these results along with Hochschild's six term exact sequence relating the first two ordinary and restricted cohomology groups to analyze the restricted cohomology group $H^2_*(\mathfrak{g})$. Our theorem states that if $\mathfrak{g}$ is a restricted simple Lie algebra, this sequence reduces to a short exact sequence relating $H^2(\mathfrak{g})$ and $H^2_*(\mathfrak{g})$. In the case that $H^2(\mathfrak{g}) = 0$, we explicitly describe the cocycles spanning $H^2_*(\mathfrak{g})$. If $H^2(\mathfrak{g}) \neq 0$, we give a procedure for describing a basis for $H^2_*(\mathfrak{g})$.

The paper is organized as follows. Section 2 gives an overview of the classification of finite dimensional simple restricted Lie algebras $\mathfrak{g}$ defined over fields of characteristic $p \geq 5$. Section 3 contains the statement and proof of the theorem relating $H^2(\mathfrak{g})$ and $H^2_*(\mathfrak{g})$ as well as an explicit description of the cocycles spanning $H^2_*(\mathfrak{g})$ when $H^2(\mathfrak{g}) = 0$. Section 4 outlines a procedure for describing a basis for $H^2_*(\mathfrak{g})$ in the case where $H^2(\mathfrak{g}) \neq 0$.

2. Restricted simple Lie algebras

Finite dimensional simple Lie algebras over fields of characteristic zero were classified more than a century ago in the work of Killing and Cartan. The well known classification theorem states that in characteristic zero, any simple Lie algebra is isomorphic to one of the linear Lie algebras A_l, B_l, C_l or D_l $(l \geq 1)$, or one of the exceptional Lie algebras E_6, E_7, E_8, F_4 or G_2. Levi's theorem implies that in characteristic 0, all extensions of a semi-simple Lie algebra split, and hence all one-dimensional central extensions of such an algebra are trivial.

The classification of modular simple Lie algebras over fields of characteristic $p \geq 5$ was completed more recently in the work by Block, Wilson, Premet and Strade in multiple papers spanning decades of research. We refer the reader to [10] for detailed account of this work. The classification states that simple Lie algebras of characteristic $p \geq 5$ fall into one of three types: *classical* Lie algebras, algebras of *Cartan type* and *Melikian algebras* (Melikian algebras are defined only for $p = 5$).