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Infinite dimensional holomorphic non-extendability and algebraic genericity

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In this note, the linear structure of the family $H_e(G)$ of holomorphic functions in a domain G of a complex Banach space that are not holomorphically continuable beyond the boundary of G is analyzed. More particularly, we prove that $H_e(G)$ contains, except for zero, a closed (and a dense) vector space having maximal dimension, as well as a maximally generated free algebra. The results obtained complete a number of previous ones by several authors.

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1. Introduction and preliminaries

In the last decade there has been a generalized trend for the search for algebraic structures inside nonlinear sets. This area of research, called *lineability* [26,36], has attracted the attention of many authors and it has been proven to be quite fruitful, with the appearance of several research papers, surveys (see, e.g. [22]), and even a monograph [5].

In this note, we focus on the family of holomorphic functions $G \to \mathbb{C}$ that cannot be holomorphically continued beyond the boundary of G, where G is a domain in a separable infinite dimensional complex Banach space E. Our aim is to contribute to complete the existing knowledge on lineability of the mentioned family.

Our notation will be rather usual. The symbols $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ will stand for the set of positive integers, the field of rational numbers, the real line and the field of complex numbers, respectively. By a domain in a complex Banach space E we mean a nonempty proper connected open subset G of E. We denote by $H(G)$ the space of all holomorphic functions $f : G \to \mathbb{C}$ (see, e.g., [24] for definitions and properties), and by ∂G the boundary of G. We say that a function $f \in H(G)$ is *holomorphically non-extendable beyond* ∂G (or that f is *holomorphic exactly* on G) whenever there do not exist two domains G_1 and G_2 in E and $\widetilde{f} \in H(G_1)$ such that

$$G_2 \subset G \cap G_1, \; G_1 \not\subset G \; \text{ and } \; \widetilde{f} = f \; \text{ on } \; G_2.$$

We denote by $H_e(G)$ the family of all $f \in H(G)$ that are holomorphic exactly on G. A domain G is called a *domain of existence* whenever $H_e(G) \neq \varnothing$. It is well known that *every* domain of $\mathbb{C}$ is a domain of existence (see [30]), but this fails for higher dimensions (see, e.g., [33]).

Now, a number of lineability concepts – that have been recently coined by a number of authors, see [6,8–10,13,15,18,23,27,29], the survey [22] and the book [5] – are in order. Namely, if X is a vector space, α is a cardinal number and $A \subset X$, then A is said to be:

- *lineable* if there is an infinite dimensional vector space M such that $M \setminus \{0\} \subset A$,
- *α-lineable* if there exists a vector space M with $\dim(M) = \alpha$ and $M \setminus \{0\} \subset A$ (hence lineability means $\aleph_0$-lineability, where $\aleph_0 = \mathrm{card}(\mathbb{N})$, the cardinality of $\mathbb{N}$), and
- *maximal lineable* in X if A is $\dim(X)$-lineable.

If, in addition, X is a topological vector space, then A is said to be:

- *dense-lineable* (*α-dense-lineable*) in X whenever there is a dense vector subspace M of X satisfying $M \setminus \{0\} \subset A$ (and $\dim(M) = \alpha$, resp.),
- *maximal dense-lineable* in X if A is $\dim(X)$-dense-lineable,
- *spaceable* (*α-spaceable*) in X if there is a closed infinite dimensional (a closed α-dimensional, resp.) vector subspace M such that $M \setminus \{0\} \subset A$, and
- *maximal spaceable* in X if A is $\dim(X)$-spaceable.