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Contractive maps on operator ideals and norm inequalities II

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Dedicated to Professor Rajendra Bhatia on his sixtyfifth birthday

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ABSTRACT

Let $(\mathcal{I}, |||.|||)$ be a norm ideal of operators equipped with a unitarily invariant norm $|||.|||$. We exploit integral representations of certain functions to prove that certain ratios of linear operators acting on operators in $\mathcal{I}$ are contractive. This leads to some new and old norm inequalities. We also lift a variety of inequalities to the operator setting, which were proved in the matrix setting earlier.

1. Introduction

Let $\mathbb{B}(\mathscr{H})$ be the algebra of all bounded linear operators on a complex separable Hilbert space $(\mathscr{H}, \langle\cdot,\cdot\rangle)$. The cone of positive operators is denoted by $\mathbb{B}(\mathscr{H})_+$. We shall

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consider a norm ideal $(\mathcal{I}, |||.|||)$ of $\mathbb{B}(\mathscr{H})$ equipped with a unitarily invariant norm and for notational convenience we shall denote throughout this by $\mathcal{I}$ instead of $(\mathcal{I}, |||.|||)$. As usual, we shall denote the operator norm by $\|\cdot\|$ and by L_X, R_Y the left, right multiplication maps on $\mathbb{B}(\mathscr{H})$ respectively, i.e. $L_X(T) = XT$ and $R_Y(T) = TY$. Since L_X and R_Y commute, we have

$$e^{L_X+R_Y}(T) = e^X T e^Y.$$

More concisely, if $f(x)$ and $g(x)$ are two infinitely many times differentiable, then we have

$$f(L_X)g(R_Y)(T) = f(X)Tg(Y).$$

Let X be selfadjoint member of $\mathbb{B}(\mathscr{H})$ and let $A \in \mathbb{B}(\mathscr{H})$ be arbitrary. Then unitary invariance of the operator norm leads to the elementary inequality

$$||A \pm iXA|| \geq ||A||.$$

The above inequality may be written as

$$||(I \pm iL_X)A|| \geq ||A||. \tag{1.1}$$

In 1995, Bhatia and Davis [6] proved a difference version of the Heinz inequality

$$|||A^\alpha XB^{1-\alpha} - A^{1-\alpha}XB^\alpha||| \leq |2\alpha - 1||||AX - XB|||, \qquad (0 \leq \alpha \leq 1) \tag{1.2}$$

for $A, B \in \mathbb{B}(\mathscr{H})_+$ and $X \in \mathcal{I}$.

Later in 1998, Kosaki [18] proved several related inequalities using Poisson integral and Fourier transform. Among those a few are matrix version of the well-known inequalities $|\sin x| \leq |x|$, $|x| \leq |\sinh x|$ and geometric–logarithmic mean inequalities.

In 2005, Jocić [13] proved Cauchy–Schwarz norm inequalities and several other related non-trivial inequalities using Gel'fand's integration techniques.

Then, in 2014, Kapil and Singh [15] proved that the function,

$$f(\alpha) = |||A^\alpha XB^{1-\alpha} - A^{1-\alpha}XB^\alpha|||$$

is monotonically decreasing for $\alpha \in [0, 1/2]$ and is monotonically increasing for $\alpha \in [1/2, 1]$, see [15, p. 491]. More precisely, it is proved that $f(\alpha)$ is a convex function on $[0, 1]$. For more such inequalities the reader may see [3,5,12,16–18,20,23,24].

To prove these and related inequalities, contractive maps on $\mathcal{I}$ play a key role. This viewpoint has been taken by some other authors. For instance, the authors in [8,9,11,18] proved positive definiteness of the functions such as $\frac{\cosh(\alpha x)}{\cosh x}$, $\frac{\sinh(\alpha x)}{\sinh x}$, for $0 < \alpha < 1$,