Linear Algebra and its Applications 513 (2017) 224–239

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Singular points of the algebraic curves associated to unitary bordering matrices

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ARTICLE INFO

Article history:
Received 25 August 2016
Accepted 19 October 2016
Available online 26 October 2016
Submitted by R. Brualdi

MSC:
15A60
14H50

Keywords:
Unitary bordering matrices
Strongly hyperbolic
Singular points
Higher rank numerical ranges

ABSTRACT

Let A be an $n \times n$ complex matrix. A ternary form associated to A is defined as the homogeneous polynomial $F_A(t, x, y) = \det(tI_n + x\Re(A) + y\Im(A))$. We prove, for a unitary boarding matrix A, the ternary form $F_A(t, x, y)$ is strongly hyperbolic and the algebraic curve $F_A(t, x, y) = 0$ has no real singular points. As a consequence, we obtain that the higher rank numerical range of a unitary boarding matrix is strictly convex.

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[1] The author is partially supported by Ministry of Science and Technology, Taiwan under MOST 104-2115-M-031-001-MY2, and the J. T. Tai & Co Foundation Visiting Research Program.
[2] The author is supported in part by Japan Society for the Promotion of Science, KAKENHI, project number 15K04890.

http://dx.doi.org/10.1016/j.laa.2016.10.018

1. Introduction

Let A be an $n \times n$ complex matrix. For an integer $1 \leq k \leq n$, the rank-k numerical range (or the higher rank numerical range) of A is defined as the set

$$\Lambda_k(A) = \{\lambda : PAP = \lambda P \text{ for some } k\text{-dimensional orthogonal projection } P\}.$$

In the case $k = 1$, $\Lambda_k(A)$ reduces to the classical numerical range of A which is defined by

$$W(A) = \{\xi^* A\xi : \xi \in \mathbf{C}^n, \xi^*\xi = 1\}.$$

Woerdeman [26] proved the convexity of $\Lambda_k(A)$ which generalizes Toeplitz–Hausdorff theorem of $W(A)$ (cf. [11,24]). The real ternary form $F_A(t, x, y)$ associated to A is defined by

$$F_A(t, x, y) = \det(tI_n + x\Re(A) + y\Im(A)),$$

where the two hermitian matrices $\Re(A) = (A + A^*)/2$, $\Im(A) = (A - A^*)/(2i)$ correspond to the Cartesian decomposition $A = \Re(A) + i\Im(A)$. Kippenhahn [14] showed that $W(A)$ is the convex hull of the real part of the dual curve $G_A(t, x, y) = 0$ of the algebraic curve $F_A(t, x, y) = 0$. The homogeneous polynomial $G_A(t, x, y)$ is irreducible in the polynomial ring $\mathbb{C}[t, x, y]$ if and only if the polynomial $F_A(t, x, y)$ is irreducible in the polynomial ring. Flat portions and sharp points on the boundary of $\Lambda_k(A)$ are deeply related to the singular points and the bi-tangents of the algebraic curve $F_A(t, x, y) = 0$ (cf. [2]).

An arbitrary real ternary form $F(t, x, y)$ of degree $n \geq 2$ is called hyperbolic with respect to $(t, x, y) = (1, 0, 0)$ if $F(1, 0, 0) \neq 0$, and the equation $F(t, x_0, y_0) = 0$ in t has n real roots counting the multiplicities for every $(x_0, y_0) \in \mathbb{R}^2$. A hyperbolic form is called strongly hyperbolic if $F(t, x_0, y_0) = 0$ in t has n distinct real roots for every $(0, 0) \neq (x_0, y_0) \in \mathbb{R}^2$. A point $P = (t_0, x_0, y_0)$ of the complex projective curve

$$C_F = \{[t, x, y] \in \mathbb{CP}^2 : F(t, x, y) = 0\}$$

is called a singular point if the gradient of $F(t, x, y)$ at P is 0, where $\mathbb{CP}^2$ denotes the complex projective plane, that is, the quotient space $\mathbb{C}^3 \setminus \{(0, 0, 0)\}/_\sim$ with respect to the relation $(t_1, x_1, y_1) \sim (t_2, x_2, y_2)$ whenever $(t_2, x_2, y_2) = \alpha(t_1, x_1, y_1)$ for a nonzero complex number α. The singular points of the algebraic curve $F_A(t, x, y) = 0$ associated to a matrix A play an important role in studying the numerical range of the matrix (cf. [3,12]).

An $n \times n$ complex matrix A is called a unitary bordering matrix (or completely non-unitary contraction with defect index 1) if A is a contraction, that is, $\langle A^*A\xi, \xi\rangle \leq \langle \xi, \xi\rangle$ for $\xi \in \mathbb{C}^n$, $\text{rank}(I_n - A^*A) = 1$ and the modulus of any eigenvalue of A is strictly less than 1. In other words, A is an $n \times n$ unitary bordering matrix if its the