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Lieb's concavity theorem, matrix geometric means, and semidefinite optimization

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A B S T R A C T

A famous result of Lieb establishes that the map $(A, B) \mapsto \mathrm{tr}\left[K^* A^{1-t} K B^t\right]$ is jointly concave in the pair (A, B) of positive definite matrices, where K is a fixed matrix and $t \in [0, 1]$. In this paper we show that Lieb's function admits an explicit semidefinite programming formulation for any rational $t \in [0, 1]$. Our construction makes use of a semidefinite formulation of weighted matrix geometric means. We provide an implementation of our constructions in Matlab.

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1. Introduction

In 1973 Lieb [19] proved the following fundamental theorem.

Theorem 1 *(Lieb). Let K be a fixed matrix in $\mathbb{C}^{n\times m}$. Then for any $t \in [0,1]$, the map*

$$(A, B) \mapsto \operatorname{tr}\left[K^* A^{1-t} K B^t\right] \tag{1}$$

is jointly concave in (A, B) where A and B are respectively $n \times n$ and $m \times m$ Hermitian positive definite matrices.

This theorem plays a fundamental role in quantum information theory and was used for example to establish convexity of the quantum relative entropy as well as strong subadditivity [21]. In this paper we give an explicit representation of Lieb's function using semidefinite programming when t is a rational number. More precisely we prove:

Theorem 2. *Let K be a fixed matrix in $\mathbb{C}^{n\times m}$ and let $t = p/q$ be any rational number in $[0,1]$. Then the convex set*

$$\left\{(A, B, \tau) : \operatorname{tr}\left[K^* A^{1-t} K B^t\right] \geq \tau\right\}$$

has a semidefinite programming representation with at most $2\lfloor \log_2 q\rfloor + 3$ linear matrix inequalities of size at most $2nm \times 2nm$.

Semidefinite programming is a class of convex optimization problems that can be solved in polynomial-time and that is supported by many existing numerical software packages. Having a semidefinite programming formulation of a function allows us to combine it with a wide family of other semidefinite representable functions and constraints, and solve the resulting problem to global optimality. In fact we have implemented our constructions in the Matlab-based modeling language CVX [14] and we are making them available online on the webpage

http://www.damtp.cam.ac.uk/user/hf323/lieb_cvx.html.

Matrix geometric means. Our proof of Theorem 2 relies crucially on the notion of *matrix geometric mean*. Given $t \in [0,1]$ and positive definite matrices A and B, the *t-weighted matrix geometric mean* of A and B denoted interchangeably by $G_t(A, B)$ or $A\#_t B$ is defined as:

$$G_t(A, B) = A\#_t B := A^{1/2}\left(A^{-1/2} B A^{-1/2}\right)^t A^{1/2}. \tag{2}$$

Note that when A and B are scalars (or commuting matrices) this formula reduces to the simpler expression $A^{1-t}B^t$. Equation (2) constitutes a generalization of the geometric