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Wreath product of matrices

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A B S T R A C T

We introduce a new matrix product, that we call the wreath product of matrices. The name is inspired by the analogous product for graphs, and the following important correspondence is proven: the wreath product of the adjacency matrices of two graphs provides the adjacency matrix of the wreath product of the graphs. This correspondence is exploited in order to study the spectral properties of the famous Lamplighter random walk: the spectrum is explicitly determined for the "Walk or switch" model on a complete graph of any size, with two lamp colors. The investigation of the spectrum of the matrix wreath product is actually developed for the more general case where the second factor is a circulant matrix. Finally, an application to the study of the uniqueness of the solution of generalized Sylvester matrix equations is treated.

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1. Introduction

A classical tool to deal with combinatorial, probabilistic and analytical problems related to finite graphs is given by the corresponding adjacency matrix, whose rows and columns are indexed by the vertices of the graph, and where the adjacency of two vertices in the graph corresponds to a nonzero entry at the intersection of the corresponding row and column. A natural question arising in this setting asks what would be the appropriate product between matrices associated with a given product of graphs. An easy example is given by the direct product of graphs, whose adjacency matrix can be represented by the Kronecker product of the corresponding adjacency matrices. Similarly, the Cartesian product and the lexicographic product of graphs correspond to the so-called crossed and nested product of matrices, respectively (see [4]). The reader can refer, for instance, to the papers [15–17] for definitions and properties of these graph products (see also the beautiful handbook [14]).

In this paper, we define an opportune and general operation between two square matrices A and B: the *wreath product* $A \wr B$. We show some interesting algebraic properties of such product, focusing our attention on the case in which B is a circulant matrix. In this case, we are able to provide a reduction formula for the spectrum of the matrix $A \wr B$. More precisely, even if the matrix $A \wr B$ has order nm^n when A has order n and B has order m, we prove that its spectrum can be explicitly determined by computing the spectrum of much smaller matrices of order n.

It turns out that the wreath product of matrices is the matrix-analogue of the classical wreath product of graphs (see, for instance, [13]): the adjacency matrix of the wreath product of the graphs $\mathcal{G}$ and $\mathcal{G}'$ is given by $A \wr B$, if A and B are the adjacency matrices of the graphs $\mathcal{G}$ and $\mathcal{G}'$, respectively. Sometimes, operations on matrices can have an interesting probabilistic interpretation, since they can be used to model Markov chains. We want to mention here the papers [4,6], where two families of Markov chains are introduced by using suitable Kronecker products, and the papers [5,7], where these models and their spectral properties are investigated in connection with random walks on trees or on more general combinatorial structures.

In our case, the connection with the probability is achieved by the notion of Lamplighter random walk. This is a well known model in the literature, and several papers have been devoted to its analysis, mainly in the case where the underlying graph is the discrete line. Spectral computations for the Lamplighter random walk and related graphs have been developed in the infinite setting [1,22], as well as in the finite setting [18,19]. In this paper, we prove that the spectral analysis of the matrix $A \wr B$ in the case in which A is the adjacency matrix of a regular graph, provides the spectrum of the Lamplighter random walk on such a graph (see also [10], where the Lamplighter random walk is studied in connection with the zig-zag product of graphs): an explicit computation is performed for the two colors model on the complete graph. In the more recent paper [12], a detailed description of the spectrum for the Lamplighter random walk, when the color graph is complete on any number of vertices, is given. This suggests the idea of