Linear Algebra and its Applications 510 (2016) 1–9

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Laplacian matrices of general complex weighted directed graphs ☆

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ARTICLE INFO

Article history:
Received 5 December 2015
Accepted 8 August 2016
Available online 10 August 2016
Submitted by R. Brualdi

MSC:
05C50
05C20
05C22

Keywords:
Complex weighted directed graphs
Laplacian matrix
Balancedness

ABSTRACT

We introduce the concept of general complex weighted directed graphs where each edge is assigned a complex number. Necessary and sufficient conditions for the Laplacian matrix to be singular/nonsingular are derived. Our results give the relationship between the Laplacian matrix and the structure of its corresponding directed graph. Compared with the existing results, our main contribution is that our results are established without the restriction that the adjacency matrix is Hermitian.

1. Introduction

A complex weighted directed graph is defined as $G = (V, E, A)$, where $V = \{1, \dots, n\}$ is the vertex set, $E \subset V \times V$ is the edge set and $A = A(G) \in \mathbb{C}^{n\times n}$ is the adjacency matrix. The weight of (i, j) is given by $a_{ji} \in \mathbb{C}$. We use the convention that $a_{ji} = 0$ if and only if $(i, j) \notin E$. We assume that $a_{ii} = 0$, i.e., G has no self-loop. Clearly, there is

☆ This work is supported by the National Natural Science Foundation of China under grant 11301114.
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http://dx.doi.org/10.1016/j.laa.2016.08.011

a one-to-one correspondence between G and its adjacency matrix A. For convenience, we also use the notation $G(A)$ to denote the weighted directed graph G induced by matrix A. Given a complex weighted directed graph $G = (V, E, A)$, define the modulus in-degree and out-degree of vertex i by $d_i^{in} = \sum_{j=1}^{n} |a_{ij}|$ and $d_i^{out} = \sum_{j=1}^{n} |a_{ji}|$, respectively. Denote by $D^{in}(G)$ and $D^{out}(G)$ the diagonal matrices where d_i^{in} and d_i^{out} are their ith diagonal entries, respectively. The Laplacian matrix $L(G)$ of G is defined as $L(G) = D^{in}(G) - A(G)$. More precisely, $L(G) = (l_{ij}) \in \mathbb{C}^{n\times n}$ is given by

$$l_{ij} = \begin{cases} -a_{ij}, & i \neq j, \\ \sum_{i=1}^{n} |a_{ij}|, & i = j. \end{cases}$$

The study of the Laplacian matrix has attracted lots of attention of researchers from different disciplines. Research on this subject can be roughly categorized by the set of values that the weights may take. When the adjacency matrix is nonnegative, the corresponding Laplacian matrix has a long history. We refer the reader to the survey papers [2,16]. The Laplacian matrix of a nonnegative weighted graph has found the broad applications in control engineering [7,8]. An extension of nonnegative weighted graphs is signed graphs where the weights of edges can be negative [14,15]. Signed graphs play a crucial role in analysis of social networks [3]. Recently, Reff [9,10] has studied complex unit gain graphs, which are defined based on undirected graphs. More precisely, each orientation of an edge of undirected graphs is given a complex unit that is the inverse of the complex unit assigned to the opposite orientation. This definition naturally gives the Hermitian adjacency matrices. We see that our definition on general complex weighted directed graphs generalizes this definition on complex unit gain graphs in the sense that they are consistent when adjacency matrix A in our definition becomes Hermitian with its entries being complex units.

The authors of [1,5] have introduced the concept of weighted directed graphs, where adjacency matrices are also Hermitian with complex units as entries. Interestingly, complex weighted directed graphs are shown to be very useful in the distributed control of multi-agent systems [6]. However, unlike in the works [1,5,9,10], adjacency matrices in [6] are not necessarily Hermitian. Hence the existing results mentioned above cannot be applied. This is our motivation to study complex weighted directed graphs with general adjacency matrices. We derive some necessary and sufficient criteria for the Laplacian matrix to be singular (or nonsingular), which relate the properties of the Laplacian matrix to the structure of the associated directed graph. In particular, it is worth mentioning that our results are obtained without assuming that the Laplacian matrix is Hermitian. This feature also implies that our results cannot be derived from general results on gain graphs in [13]. We study the properties of the Laplacian matrix in Section 2. Section 3 gives the proof of Theorem 8.

We end this section with some definitions to be used in this paper. The *weight* of a (directed) cycle is defined as the product of weights on all its edges. A cycle is said to be *positive* if it has a positive weight. A digraph is said to be *balanced* if all of its cycles are