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2-Walk-regular graphs with a small number of vertices compared to the valency

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ABSTRACT

In 2013, it was shown that, for a given real number $\alpha > 2$, there are only finitely many distance-regular graphs Γ with valency k and diameter $D \geq 3$ having at most αk vertices, except for the following two cases: (*i*) $D = 3$ and Γ is imprimitive; (*ii*) $D = 4$ and Γ is antipodal and bipartite. In this paper, we will generalize this result to 2-walk-regular graphs. In this case, also incidence graphs of certain group divisible designs appear.

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1. Introduction

For unexplained terms, see the next section.

In 2013, Koolen and Park [11] studied distance-regular graphs with a small number of vertices relative to their valency and they have shown the following result.

Theorem 1.1. *(Cf. [11, Theorem 1]) Let $\alpha > 2$ be a real number. Then there are finitely many distance-regular graphs Γ with valency k, diameter $D \geq 3$ and v vertices satisfying $v \leq \alpha k$ unless one of the following holds:*

i) $D = 3$ and Γ is imprimitive,
ii) $D = 4$ and Γ is antipodal and bipartite.

Note that a t-walk-regular graph is a common generalization of distance-regular graphs on the one hand and t-arc-transitive graphs on the other hand. We focus on the case where $t = 2$ because s-walk-regular graphs are t-walk-regular if $s \geq t$ and many properties of distance-regular graphs can be generalized to 2-walk-regular graphs. In this paper we use some results (Lemma 2.2 and Lemma 2.7) that hold for 2-walk-regular graphs but are not true in general for 1-walk-regular graphs as shown in [5]. We will generalize Theorem 1.1 to the class of 2-walk-regular graphs as follows.

Theorem 1.2. *Let $\alpha > 2$ be a real number and let Γ be a 2-walk-regular graph with valency k, diameter $D \geq 3$ and v vertices. If $v \leq \alpha k$, then one of the following holds:*

i) $k \leq C(\alpha)$, and hence $v \leq C(\alpha)\alpha$, where $C(\alpha) = \frac{(\alpha^{11}-1)(\alpha^{11}+2)}{2}$;
ii) $D = 3$ and Γ is an imprimitive distance-regular graph;
iii) $D = 4$ and Γ is a bipartite antipodal distance-regular graph;
iv) $D = 4$ and Γ is the incidence graph of an $(n, m; k; 0, \lambda_2)$-group divisible design with the dual property, where $v = 2nm$.

Note that there are infinitely many 2-walk-regular graphs with $v = 4k + 4$ that are not distance-regular, satisfying *iv*) (for example, see Remark 3.2). And this result shows that although the class of 2-walk-regular graphs is much larger than the class of distance-regular graphs, 2-walk-regular graphs behave fairly similar as distance-regular graphs. But the proof of Theorem 1.2 needs some more ideas, as for 2-walk-regular graphs you loose quite a bit of the combinatorial properties of distance-regular graphs.

We will also show that a proper 2-walk-regular graph with valency k has at least $3k + 2$ vertices. Moreover, if it is bipartite, then it has at least $4k + 2$ vertices. We will also give an infinite family of bipartite proper 2-walk-regular graphs with valency k and $4k + 4$ vertices. Park [13] gave the classification of primitive distance-regular graphs with at most $3k + 1$ vertices. We do not know of any proper 2-walk-regular graph with valency k and at most $4k + 3$ vertices.