Linear Algebra and its Applications 510 (2016) 42–63

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Complementarity properties of singular M-matrices

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ARTICLE INFO

Article history:
Received 24 February 2016
Accepted 1 August 2016
Available online 5 August 2016
Submitted by R. Brualdi

MSC:
15A09
90C33

Keywords:
M-matrix with "property c"
Group inverse
Range monotonicity
Strictly range semimonotonicity
Range column sufficiency
$P_{\#}$-matrix
Linear complementarity problem

ABSTRACT

For a matrix A whose off-diagonal entries are nonpositive, its nonnegative invertibility (namely, that A is an invertible M-matrix) is equivalent to A being a P-matrix, which is necessary and sufficient for the unique solvability of the linear complementarity problem defined by A. This, in turn, is equivalent to the statement that A is strictly semimonotone. In this paper, an analogue of this result is proved for singular symmetric Z-matrices. This is achieved by replacing the inverse of A by the group generalized inverse and by introducing the matrix classes of strictly range semimonotonicity and range column sufficiency. A recently proposed idea of $P_{\#}$-matrices plays a pivotal role. Some interconnections between these matrix classes are also obtained.

1. Introduction

$\mathbb{R}^{n\times n}$ denotes the space of all real square matrices of order n and $\mathbb{R}^n$ denotes the real Euclidean space of real vectors with n coordinates. For $x \in \mathbb{R}^n$, we write $x \geq 0$ to denote

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http://dx.doi.org/10.1016/j.laa.2016.08.003

that all the coordinates of x are nonnegative. This is written as $x \in \mathbb{R}^n_+$, where $\mathbb{R}^n_+$ is the nonnegative orthant of $\mathbb{R}^n$. $x > 0$ signifies the fact that all the coordinates of x are positive. A real matrix B is said to be *nonnegative* if all its entries are nonnegative. This is denoted by $B \geq 0$. One of the central objects of interest in this work is the concept of a linear complementarity problem, which we discuss next. For $x, y \in \mathbb{R}^n$, we use $\langle x, y\rangle$ to denote the inner product $x^T y$ and $x \circ y$ to denote the Hadamard entrywise product of x and y. Let $A \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$. The *linear complementarity problem* $LCP(A, q)$ is to determine if there exists $x \in \mathbb{R}^n$ such that $x \geq 0$, $y = Ax + q \geq 0$ and $\langle y, x\rangle = 0$. If such a vector x exists, then $LCP(A, q)$ is said to have a *solution*. $SOL(A, q)$ denotes the set of all solutions of $LCP(A, q)$. Various classes of matrices have been introduced to study the existence and uniqueness of solutions of $LCP(A, q)$. Let us recall some of the relevant ones. A real square matrix A is called a *P-matrix* if all its principal minors are positive. It is well known that A is a P-matrix if and only if the implication

$$x \circ Ax \leq 0 \Longrightarrow x = 0$$

holds [3]. A famous result in the theory of linear complementarity problems states that $LCP(A, q)$ has a unique solution for all $q \in \mathbb{R}^n$ if and only if A is a P-matrix [3]. Let us consider the second class of matrices. A real square matrix A is said to be a *strictly semimonotone* matrix if

$$x \geq 0 \text{ and } x \circ Ax \leq 0 \Longrightarrow x = 0.$$

It is well known that A is a strictly semimonotone matrix if and only if $LCP(A, q)$ has a unique solution for all $q \in \mathbb{R}^n_+$ (Theorem 3.9.11) [3]. Any P-matrix is a strictly semimonotone matrix, while the converse could be shown to be false. However, these two classes coincide for a matrix class which we consider next. A is called a *Z-matrix*, if all its off-diagonal entries are nonpositive. Note that if A is a Z-matrix, then $A = sI - B$, for some $s \in \mathbb{R}$ with $s > 0$ and $B \geq 0$. A Z-matrix A is called an *M-matrix* if in the representation as above, one also has $s \geq \rho(B)$, where $\rho(B)$ denotes the spectral radius of B. For a Z-matrix A to be a P-matrix, more than fifty characterizations are proved in the literature. We refer to the excellent book [2], for these. In what follows, we list out the conditions that are pertinent to the discussion here.

Theorem 1.1. *[2,12] Let $A \in \mathbb{R}^{n\times n}$ be a Z-matrix. Then the following statements are equivalent:*

(a) *A is a P-matrix.*
(b) *A^{-1} exists and $A^{-1} \geq 0$.*
(c) *A is an invertible M-matrix.*
(d) *A is a strictly semimonotone matrix.*