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Left and right generalized inverses

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A B S T R A C T

This article examines a way to define left and right versions of the large class of "(b, c)-inverses" introduced by the writer in (2012) [6]: Given any semigroup S and any $a, b, c \in S$, then a is called *left* (b, c)*-invertible* if $b \in Scab$, and $x \in S$ is called a *left* (b, c)*-inverse* of a if $x \in Sc$ and $xab = b$, and dually $c \in cabS$, $z \in Sb$ and $caz = z$ for right (b, c)-inverses z of a. It is shown that left and right (b, c)-invertibility of a together imply (b, c)-invertibility, in which case every left (b, c)-inverse of a is also a right (b, c)-inverse, and conversely, and then all left or right (b, c)-inverses of a coincide. When $b = c$ (e.g. for the Moore-Penrose inverse or for the pseudo-inverse of the author) left (b, b)-invertibility coincides with right (b, b)-invertibility in every strongly π-regular semigroup. A fundamental result of Vaserstein and Goodearl, which guarantees the left-right symmetry of Bass's property of stable range 1, is extended from two-sided inverses to left or right inverses, and, for central b, to left or right (b, b)-inverses.

1. Introduction

In any semigroup S (or, in particular, any associative ring or algebra) with unit element 1, and for any given $a \in S$, the properties $1 \in Sa$ [resp. $1 \in aS$] of left [resp.

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right] invertibility are often useful (e.g. in defining Bass's property of stable range 1, see Section 4) as weaker versions of ordinary two-sided invertibility, and it is natural to seek corresponding one-sided versions for at least some types of generalized invertibility. However, apparently not much has yet been done in this direction, presumably mainly because, for all the various specific two-sided generalized inverses y of a which are most widely used or discussed (such as the Moore-Penrose inverse $y = a^\dagger$ [12] and the pseudo-inverse $y = a'$ introduced by the present writer [5]), it is not clear how to modify their original (or other known) definitions so as to yield a viable corresponding left or right version.

This article had its beginning in the observation that at least this initial barrier to progress can be circumvented by using the "(b, c)-inverses" introduced by the present writer in [6, p. 1911, Definition 1.3]. For the reader's convenience, we recall how these are defined:

Definition 1.1. Let S be any semigroup and let $a, b, c \in S$. Then a is called (b, c)-*invertible* if there exists $y \in S$ such that

$$y \in bSy \cap ySc$$

and

$$yab = b, \quad cay = c.$$

Any such y is called a (b, c)-*inverse* of a.

For the classical inverse $y = a^{-1}$, defined as usual by $ya = ay = 1$, just take $b = c = 1$. As discussed in [6, p. 1910], by choosing b and c appropriately we obtain equivalent alternative definitions of most other known generalized inverses (e.g. $b = c = a^*$ for $y = a^\dagger$, while $b = c = a^j$ for suitable $j \in \mathbb{N}$ gives $y = a'$). Moreover, y is always unique when it exists, so that we can then call y the (b, c)-inverse of a.

While there are different ways one might choose to formulate a definition of what a left or right (b, c)-inverse (or left or right (b, c)-invertibility) should be, in order to get satisfactory consequences from the least restrictive assumptions it seems that the most rewarding is as follows (suggested by [6, p. 1912, Theorem 2.2]):

Definition 1.2. Let S be any semigroup and let $a, b, c \in S$. Then we shall say that a is *left* (b, c)-*invertible* if $b \in Scab$, or equivalently if there exists $x \in Sc$ such that $xab = b$, in which case any such x will be called a *left* (b, c)-*inverse* of a.

Dually, a is called *right* (b, c)-*invertible* if $c \in cabS$, or equivalently if there exists $z \in bS$ such that $caz = c$, and any such z will be called a *right* (b, c)-*inverse* of a.

In particular, by taking $(b, c) = (a^*, a^*)$ or (a^j, a^j) etc., we immediately obtain corresponding definitions of left and right Moore-Penrose inverses or pseudo-inverses etc. However, these are not very interesting examples of left or right (b, c)-inverses, since in