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Automorphisms of the endomorphism algebra of a free module

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ABSTRACT

Let R be a commutative ring with identity $1 \in R$ and V a free R-module of arbitrary rank. Let $End_R(V)$ denote the R-algebra of all R-linear endomorphisms of V. We show that all R-algebra automorphisms of $End_R(V)$ are inner if R is a Bezout domain. We also consider 2-local automorphisms of $End_R(V)$.

1. Introduction

Let A and B be central simple algebras over the field F. The celebrated Skolem–Noether Theorem goes back to [11] and [10] and states that any two algebra homo-

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morphisms $\varphi, \psi : A \to B$ are conjugate, i.e. there exists a unit $b \in B$ such that $\varphi(x) = b\psi(x)b^{-1}$ for all $x \in A$. It is an immediate consequence of this theorem that all automorphisms of the algebra $Mat_{n\times n}(F)$, the ring of $n \times n$ matrices over F, are inner automorphisms. This result has been extended to many more classes of algebras. For example, it was shown in [6] that all automorphisms of $M = Mat_{n\times n}(R)$ are inner, if R is a unique factorization domain (UFD). Moreover, for any (!) commutative ring, the group of outer automorphisms $Out(M) = Aut(M)/Inn(M)$ is abelian and bounded by n. For Dedekind domains R, [6] also contains a nice description of $Out(M)$ in terms of the class group of R. Since there exist non-Noetherian Bezout domains R, which are not UFD, we are able to show a little bit more:

- If R is a Bezout domain, then all automorphisms of $Mat_{n\times n}(R)$ are inner.

The paper [6, Example 6] also contains an example of some $\tau \in Aut(Mat_{2\times 2}(\mathbb{Z}[\sqrt{-5}\,]))$ that is not inner. We will construct a similar example pointing out how maximal, non-principal ideals of the Dedekind domain come into play.

Let R be a commutative ring with identity $1 \in R$ and V some free R-module. Then Isaac's result can be stated as: If R is a UFD and V has **finite** rank, then all automorphisms of $End_R(V)$ are inner. Let $Fin_R(V) = \{\varphi \in End_R(V) : \varphi(V)$ is contained in a finitely generated submodule of $V\}$. In his classic monograph [7, Isomorphism Theorem, page 79] Jacobson shows that if A is a subalgebra of $End_D(V)$ containing $Fin_D(V)$, where D is a division ring and V a D-vector space of arbitrary dimension, then any automorphism of A is induced (via conjugation) by some semi-linear map on V. We will extend this result by replacing D by suitable commutative rings. We will show:

- Let R be a Bezout domain and V any free R-module. If A is a subalgebra of $End_R(V)$ with $Fin_R(V) \subseteq A$, then all automorphisms of A are restrictions of some inner automorphism of $End_R(V)$.

There is a large amount of literature on automorphisms of certain finite dimensional algebras, see, for example, [2], [4] or [9]. Most similar results on infinite dimensional algebras all seem to be connected in some way to analysis, like operator algebras on Banach spaces or C^*-algebras.

Let A be an algebra over the commutative ring R and $\varphi : A \to A$ an R-linear map. Moreover, let $n \in \mathbb{N}$. We call φ a n-local automorphism if for any n elements $x_i \in A$ there is some $\alpha \in Aut(A)$ such that $\varphi(x_i) = \alpha(x_i)$ for all $1 \leq i \leq n$. If $n = 1$, then φ is called a local automorphism. While any n-local automorphism is clearly injective, surjectivity does not follow in general. In a forthcoming paper [1] we will present examples of fields F and F-algebras A for which there exists an F-linear map $\varphi : A \to A$ which is an n-local automorphism for each $n \geq 1$, but φ is not surjective and thus not an automorphism.