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$\{0,1\}$ completely positive tensors and multi-hypergraphs

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A B S T R A C T

Completely positive graphs have been employed to associate with completely positive matrices for characterizing the intrinsic zero patterns. As tensors have been widely recognized as a higher-order extension of matrices, the multi-hypergraph, regarded as a generalization of graphs, is then introduced to associate with tensors for the study of complete positivity. To describe the dependence of the corresponding zero pattern for a special type of completely positive tensors—the $\{0,1\}$ completely positive tensors, the completely positive multi-hypergraph is defined. By characterizing properties of the associated multi-hypergraph, we provide necessary and sufficient conditions for any $(0,1)$ associated tensor to be $\{0,1\}$ completely positive. Furthermore, a necessary and sufficient condition for a uniform multi-hypergraph to be a completely positive multi-hypergraph is proposed as well.

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1. Introduction

Completely positive matrices (cp matrices) [6,28], as a special type of nonnegative matrices, have wide applications in combinatorial theory including the study of block designs [16], and in optimization especially in creating convex formulations of NP-hard problems, such as the quadratic assignment problem in combinatorial optimization and the polynomial optimization problems [1–3,17,31]. The verification of cp matrices is generally NP-hard unless for small scale matrices. For example, all $n \times n$ nonnegative symmetric positive semidefinite matrices (usually called the doubly nonnegative (dnn) matrices) are cp-matrices whenever $n \leq 4$ [7]. For general case, it is obvious that cp matrices are dnn, but not always true conversely [4,13,27]. It depends on some inherited zero pattern which cp matrices possess. To describe this dependence, the tool of graphs was employed and the completely positive graph (cp graph) was introduced which has all its nonnegative associated matrices being cp. Among all those properties on cp-graphs, one of the most important and well-known is a graph to be completely positive if and only if it does not have an odd cycle of length greater than 4 [19]. This gives us a very efficient way to verify cp-matrices in terms of cp graphs.

Recently, the concept of cp matrix has been extended to the higher order cp tensor, which admits its definition in a pretty natural way as initiated by Qi et al. in [25]. Analog to the matrix case, the cp tensors were employed to reformulate polynomial optimization problems [24]. Numerical optimization for the best fit of completely positive tensors with given length of decomposition was formulated as a nonnegative constrained least-squares problem in Kolda's paper [18]. For the verification of cp tensors, an efficient approach in terms of truncated moment sequences for checking completely positive tensors was proposed and an optimization algorithm based on semidefinite relaxation for completely positive tensor decomposition was established by Fan and Zhou in [14]. This approach was later accelerated with some preprocessing steps by Luo and Qi in [21]. Some structured and geometrical properties on general cp tensors were also discussed in [21,26].

Inspired by the technique of using cp graphs for the characterization of cp matrices, we employ the multi-hypergraph as a tool to describe the inherited zero pattern for cp tensors, which can further assist with the verification of cp-tensors. Multi-hypergraphs appeared in the literature at least in 1988 or even earlier. Here we use the definitions in [23]. Due to complexity of cp-tensors for general higher order cases, we will focus on a special type of cp-tensors called the $\{0,1\}$ cp tensor, which is exactly a higher order extension of the $\{0,1\}$ cp matrix that has been well studied in [8,9] motivated by the applications in many fields such as the pattern recognition [20]. In order to verify $\{0,1\}$ cp tensors, we first build up the correspondence between multi-hypergraphs and symmetric tensors which are called the associated tensors. The $(0,1)$ associated tensor is also defined which is uniquely determined by the corresponding multi-hypergraph. Based on the aforementioned one-to-one relationship, we establish the necessary and sufficient conditions for a $(0,1)$ associated tensor to be $\{0,1\}$ cp in terms of some structure property