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Constrained $(0,1)$-matrix completion with a staircase of fixed zeros

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A B S T R A C T

We study the $(0,1)$-matrix completion with prescribed row and column sums wherein the ones are permitted in a set of positions that form a Young diagram. We characterize the solvability of such $(0,1)$-matrix completion problems via the nonnegativity of a structure tensor which is defined in terms of the problem parameters: the row sums, column sums, and the positions of fixed zeros. This reduces the exponential number of inequalities in a direct characterization yielded by the max-flow min-cut theorem to a polynomial number of inequalities. The result is applied to two engineering problems arising in smart grid and real-time systems, respectively.

1. Introduction

Given are two nonnegative integral vectors $r = [r_1 \quad r_2 \quad \dots \quad r_m]'$ and $h = [h_1 \quad h_2 \quad \cdots \quad h_n]'$. Assume that $\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} h_j$. Let $\mathscr{A}(r,h)$ denote the set of

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$m \times n$ $(0,1)$-matrices with row sum vector given by r and column sum vector given by h. In other words, a matrix A belongs to $\mathscr{A}(r,h)$ if and only if all its entries are 0's or 1's such that the ith row sum is r_i and the jth column sum is h_j. The completion of such $(0,1)$-matrices with given row and column sums has been attracting continuing interest since the independent pioneering works by Gale [12] and Ryser [20] in 1957. Let A^* be a $(0,1)$-matrix with row sum vector given by r such that the 1's are put as far to the left as possible. The column sum vector of A^*, denote by r^*, is referred to as the conjugate vector of r. Note that r^* naturally has its elements ordered non-increasingly. The famous Gale–Ryser Theorem says that $\mathscr{A}(r,h) \neq \emptyset$ if and only if the majorization relation $h \prec r^*$ holds, which by definition means

$$\sum_{j=1}^{k} h_j^{\downarrow} \leq \sum_{j=1}^{k} r_j^*, \text{ for } k = 1, 2, \ldots, n,$$

where $h^{\downarrow}$ stands for the reordered version of h with its elements rearranged in a non-increasing order. For a comprehensive treatment of majorization theory and its applications in combinatorics, one can refer to the book [17].

Gale and Ryser's papers pointed to a number of research opportunities. Efforts have been devoted to investigating various types of constrained $(0,1)$-matrix completion problems. In particular, attention has been paid to scenarios where the 1's are only permitted in a certain set of positions, and the remaining positions are forced to be filled with 0's. The positions wherein the 1's are permitted are called free positions, while the others are called forbidden positions. Studying the constrained $(0,1)$-matrix completion problems, in addition to being theoretically interesting, helps in many application areas such as operation research [12], discrete tomography [14], real-time systems [6], smart grid [8,13,19], electoral seat allocation [16], etc. The article [2] and the book [3] and the references therein provide a comprehensive survey. To better understand the state of the art, some pertinent results closely related to this paper are briefly reviewed below.

First, it has been widely recognized that there exists a natural one–one correspondence between a $(0,1)$-matrix and an associated network flow [12,10,2,4]. As such, any $(0,1)$-matrix completion problem, whether constrained or not, can be computationally solved via an associated maximal flow problem in polynomial time. While a numerical solution may serve the purpose in some applications, in many others an analytic characterization for the solvability of the constrained $(0,1)$-matrix completion problems is also of great interest. In general, a direct application of the well-known max-flow min-cut theorem yields a characterization given by an exponential number of inequalities [18]. It is often desirable to obtain a simpler characterization involving fewer inequalities by exploiting the underlying pattern of the fixed zeros.

Indeed, substantial progress has been made in various special cases when the fixed 0's admit certain particular structures. Reference [11] considered a subset of $\mathscr{A}(r,h)$ consisting of square $(0,1)$-matrices with zero trace, i.e., the diagonal elements are fixed to be 0's. To deal with this case, let $A^\flat$ be a zero-trace $(0,1)$-matrix with row sum vector