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Rank nonincreasing linear maps preserving the determinant of tensor product of matrices ☆

M. Antónia Duffner [a,*], Henrique F. da Cruz [b]

[a] Faculdade de Ciências da Universidade de Lisboa, Bloco C6, Piso 2, Campo Grande, 1700-016 Lisboa, Portugal

[b] Departamento de Matemática da Universidade da Beira Interior, Rua Marquês D'Avila e Bolama, 6201-001 Covilhã, Portugal

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Let $l, m_1, m_2, \dots m_l \geq 2$ be positive integers. We describe some linear maps $\phi : M_{m_1 \dots m_l}(\mathbb{F}) \to M_{m_1 \dots m_l}(\mathbb{F})$ satisfying

$$\det(\phi(A_1 \otimes \dots \otimes A_l)) = \det(A_1 \otimes \dots \otimes A_l),$$

for all $A_k \in M_{m_k}(\mathbb{F})$, $k = 1, \dots, l$.

1. Introduction

Let $M_n(\mathbb{F})$ be the linear space of n-square matrices over a field $\mathbb{F}$. For $A \in M_m(\mathbb{F})$ and $B \in M_n(\mathbb{F})$, we denote by $A \otimes B \in M_{mn}(\mathbb{F})$ their tensor product (also known as

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* Corresponding author.
E-mail addresses: mamonteiro@cii.fc.ul.pt (M.A. Duffner), hcruz@ubi.pt (H.F. da Cruz).

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the Kronecker product). Throughout this note, O_n denotes the n by n null matrix, and $E_{i,j}^{(n)}$ denotes the n by n matrix whose all entries are equal to zero except for the (i,j)-th entry which is one. The rank of a matrix A is denoted by $\rho(A)$.

In a wide variety of pure or applied studies the tensor product of matrices plays a fundamental role. For instance, in quantum physics, the quantum states of a system with n physical states are represented as n by n positive semi-definite matrices with trace one, and if A and B are two states of two quantum system, then $A \otimes B$ describes the joint (bipartite) system. Recent studies in quantum information theory require the description of linear maps that preserve certain properties of tensor product of matrices (see [3]). Therefore, many papers have been appearing in the literature with the characterization of linear maps that preserve certain properties of the tensor product of matrices, such as norms, the rank, the spectrum, the spectral radius, the numerical radius or idempotency (see [1,2,4–7]).

Let $l, m_1, m_2, \ldots m_l \geq 2$ be positive integers. Our main goal is to describe linear maps $\phi : M_{m_1 \ldots m_l}(\mathbb{F}) \to M_{m_1 \ldots m_l}(\mathbb{F})$ which satisfy

$$\det(\phi(A_1 \otimes \ldots \otimes A_l)) = \det(A_1 \otimes \ldots \otimes A_l), \tag{1}$$

for all $A_k \in M_{m_k}(\mathbb{F})$, $k = 1, \ldots, l$, with the assumption that ϕ do not increase the rank of any matrix $A_1 \otimes \ldots \otimes A_l$.

2. Main result and proofs

We call a linear map π on $M_{m_1 \ldots m_l}(\mathbb{F})$ *canonical*, if

$$\pi(A_1 \otimes \ldots \otimes A_l) = \psi_1(A_1) \otimes \ldots \otimes \psi_l(A_l),$$

for all $A_k \in M_{m_k}(\mathbb{F})$, $k = 1, \ldots, l$, where $\psi_k : M_{m_k}(\mathbb{F}) \to M_{m_k}(\mathbb{F})$, $k = 1, \ldots, l$, is either the identity map, $\psi_k(X) = X$ or the transposition map $\psi_k(X) = X^T$. In this case, we write $\pi = \psi_1 \otimes \ldots \otimes \psi_l$.

Our main theorem reads as follows:

Theorem 1. *Let $\phi : M_{m_1 \ldots m_l}(\mathbb{F}) \to M_{m_1 \ldots m_l}(\mathbb{F})$ be a linear map which does not increase the rank of any matrix $A_1 \otimes \ldots \otimes A_l \in M_{m_1 \ldots m_l}(\mathbb{F})$. Then, ϕ satisfies (1) if and only if there are invertible matrices $U, V \in M_{m_1 \ldots m_l}(\mathbb{F})$ with $\det(UV) = 1$, and a canonical map π on $M_{m_1 \ldots m_l}(\mathbb{F})$ such that*

$$\phi(A_1 \otimes \ldots \otimes A_l) = U\pi(A_1 \otimes \ldots \otimes A_l)V,$$

for all $A_k \in M_{m_k}(\mathbb{F})$, $k = 1, \ldots, l$.

For proving our main results we will need some auxiliary results.

In a recent work, [6], Lim described the linear maps that preserve the rank of the tensor product of matrices over an arbitrary field $\mathbb{F}$: