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Maximal simultaneously nilpotent sets of matrices over antinegative semirings

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We study the simultaneously nilpotent index of a simultaneously nilpotent set of matrices over an antinegative commutative semiring S. We find an upper bound for this index and give some characterizations of the simultaneously nilpotent sets when this upper bound is met. In the special case of antinegative semirings with all zero divisors nilpotent, we also find a bound on the simultaneously nilpotent index for all nonmaximal simultaneously nilpotent sets of matrices and establish their cardinalities in case of a finite S.

1. Introduction

As semirings allow for more general and joint solutions to various problems, while reducing the time complexity of existing algorithms, they are a very active research area in computer science. This algebraic structure has properties that are quite different from other classical algebraic structures as groups, rings and fields. Thus, over last decades,

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many developments in mathematics have been devoted to the research of semirings. A particular class of canonically ordered semirings (also called dioids) come rather naturally into play in connection with algebraic models for many problems arising from computer science, such as scheduling, network analysis, pathfinding problems in graphs, hierarchical clustering, parsing,...

Nilpotent matrices play a crucial role when studying matrices over semirings. In this work, we continue studying the simultaneous nilpotence of a set of matrices. The definition of the simultaneous nilpotence originates in the study of infinite products of matrices and their convergence to the zero matrix.

A *semiring* is a set S equipped with binary operations $+$ and $\cdot$ such that $(S, +)$ is a commutative monoid with identity element 0 and $(S, \cdot)$ is a monoid with identity element 1. In addition, operations $+$ and $\cdot$ are connected by distributivity and 0 annihilates S. For a semiring S we denote by $\mathcal{N}(S) \subseteq S$ the set of all nilpotent elements in S and by $Z(S)$ the set of zero-divisors in S.

A semiring is *commutative* if $ab = ba$ for all $a, b \in S$. A semiring S is *antinegative* (sometimes also called a *zerosumfree* semiring or an *antiring*), if the condition $a + b = 0$ implies that $a = b = 0$ for all $a, b \in S$.

For example, $\mathbb{Z}_2$ and Boolean algebra $\mathcal{B} = (\{0, 1\}, \vee, \wedge)$ are the smallest nontrivial semirings, with the difference that $\mathcal{B}$ is an antinegative semiring, while $\mathbb{Z}_2$ is actually a ring. The set of nonnegative integers with the usual operations of addition and multiplication is a commutative antinegative semiring. Distributive lattices and fuzzy algebras are commutative antinegative semirings.

The set $M_n(S)$ of $n \times n$ matrices over a semiring S is a semiring as well. Let $E_{i,j}$ denote the zero-one matrix with the only nonzero element in the (i, j)-th position. For a set of matrices $\mathcal{R} \subseteq M_n(S)$ we denote the set of products of k matrices by

$$\mathcal{R}^k = \{A_1 A_2 \ldots A_k;\ A_i \in \mathcal{R}\}.$$

For $1 \leq i < j \leq k$, the product $A_i A_{i+1} \ldots A_j$ is called a *subproduct* of $A_1 A_2 \ldots A_k$. We say that the set of matrices $\mathcal{R}$ is *simultaneously nilpotent* if $\mathcal{R}^p = \{0\}$ for some positive integer p. By $h(\mathcal{R})$ we denote the smallest integer p such that $\mathcal{R}^p = \{0\}$ and call it the *simultaneously nilpotent index*.

In [1] the first properties of the simultaneously nilpotent set of fuzzy matrices are given. In [2] the authors characterize the simultaneous nilpotence of fuzzy matrices and extend some of the results to the bounded distributive lattices. Further properties of nilpotent fuzzy matrices were investigated in [3]. In [5] the authors give some sufficient and some necessary conditions on elements of $\mathcal{R} \subseteq M_n(S)$ for an antinegative semiring S, that cause $\mathcal{R}$ to be simultaneously nilpotent. Tan [4] proves some characterizations of simultaneously nilpotent matrices over commutative antirings with additional assumption of S being without zero-divisors, $\mathcal{N}(S)$ being simultaneously nilpotent, or when $\mathcal{N}(S) = Z(S)$.