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Limiting behavior of immanants of certain correlation matrix

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ABSTRACT

A correlation matrix is a positive semi-definite Hermitian matrix with all diagonals equal to 1. The minimum of the permanents on singular correlation matrices is conjectured to be given by the matrix Y_n, all of whose non-diagonal entries are $-1/(n-1)$. Also, Frenzen–Fischer proved that $\operatorname{per} Y_n$ approaches to $e/2$ as $n \to \infty$. In this paper, we analyze some immanants of Y_n, which are the generalizations of the determinant and the permanent, and we generalize these results to some other immanants and conjecture most of those converge to 1.

1. Introduction

Let χ be a character of a subgroup G of the symmetric group $\mathfrak{S}_n$, and $A = (a_{ij})$ an $n \times n$ complex matrix. The generalized matrix function associated with G and χ is defined to be

$$d_\chi^G(A) = \sum_{\sigma \in G} \chi(\sigma) a_{1\sigma(1)} \cdots a_{n\sigma(n)}.$$

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When $G = \mathfrak{S}_n$ and χ is its irreducible character, d_χ^G is called an immanant. When $\chi = \mathrm{sgn}$, the immanant $d_{\mathrm{sgn}}^{\mathfrak{S}_n}$ is the determinant, and when χ is trivial, $d_{\mathrm{triv}}^{\mathfrak{S}_n}$ is the permanent. Since there is a one-to-one correspondence between Young diagrams of n boxes and irreducible representations of $\mathfrak{S}_n$, we simply denote the immanant associated with the character corresponding to the Young diagram λ by d_λ. We also define the normalized generalized matrix function $\overline{d}_\chi^G$ to be $\overline{d}_\chi^G = d_\chi^G/\chi(\mathrm{id})$.

A correlation matrix is a positive semi-definite Hermitian matrix with all diagonals equal to 1. The minimum of the permanents on $n \times n$ singular correlation matrices is conjectured by Pierce [7] to be given by the matrix Y_n, all of whose non-diagonal entries are $-1/(n-1)$. Motivated by this conjecture, Frenzen–Fischer [2] showed that the sequence $\{\operatorname{per} Y_n\}$ is monotonically decreasing for $n \geq 2$ and

$$\lim_{n\to\infty} \operatorname{per} Y_n = \frac{e}{2}. \tag{1}$$

It is easy to see that $\det Y_n = 0$ for all $n \geq 2$. One can notice that the permanent and the determinant are the (normalized) immanants corresponding to (n) and (1^n), respectively. Thus, we shall be interested in the limiting behavior of other normalized immanants of Y_n.

In Section 2, we find the limits of the determinantal and permanental minors of Y_n, which are the key lemmas in this paper. In Section 3 and 4, the immanants associated with hook Young diagrams $(k, 1^{n-k})$ and other Young diagrams are discussed using Littlewood–Richardson's correspondence between Schur functions and immanants. Notice that the number of boxes of Young diagrams increases as $n \to \infty$, so that we describe the behavior of immanants in terms of limit shapes of diagrams. In light of the results, we conjecture that the limits of immanants for many cases converge to 1:

Conjecture 1. *Let $\{\lambda^{(n)}\}$ be a sequence of Young diagrams such that $|\lambda^{(n)}| = n$ and*

$$\lambda^{(1)} \subset \lambda^{(2)} \subset \lambda^{(3)} \subset \cdots,$$

and let $\mu^{(n)}$ be the conjugate of $\lambda^{(n)}$. If $\lim_{n\to\infty} \lambda_1^{(n)}/n = 0$ and $\lim_{n\to\infty} \mu_1^{(n)}/n = 0$, then

$$\lim_{n\to\infty} \overline{d}_{\lambda^{(n)}}(Y_n) = 1.$$

In Section 5, we point out some remarks on the permanental dominance conjecture, which can be the motivation to find the immanants of Y_n. In particular, we conjecture that Y_n maximizes $(\overline{d}_\lambda(A) - \det A)/(\operatorname{per} A - \det A)$ except for $\lambda = (n-1, 1)$.

2. Lemmas on principal minors of Y_n

Let I_n be the $n \times n$ identity matrix and J_n the $n \times n$ matrix with all entries equal to 1. We introduce two formulae for the determinant and the permanent for later use.