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# Real eigenvalue statistics for products of asymmetric real Gaussian matrices

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Random matrices formed from i.i.d. standard real Gaussian entries have the feature that the expected number of real eigenvalues is non-zero. This property persists for products of such matrices, independently chosen, and moreover it is known that as the number of matrices in the product tends to infinity, the probability that all eigenvalues are real tends to unity. We quantify the distribution of the number of real eigenvalues for products of finite size real Gaussian matrices by giving an explicit Pfaffian formula for the probability that there are exactly $k$ real eigenvalues as a determinant with entries involving particular Meijer $G$-functions. We also compute the explicit form of the Pfaffian correlation kernel for the correlation between real eigenvalues, and the correlation between complex eigenvalues. The simplest example of these — the eigenvalue density of the real eigenvalues — gives by integration the expected number of real eigenvalues. Our ability to perform these calculations relies on the construction of certain skew-orthogonal polynomials in the complex plane, the computation of which is carried out using their relationship to particular random matrix averages.



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## 1. Introduction

A basic question in random matrix theory is to ask for the probability distribution of the number of real eigenvalues for an ensemble of $N \times N$ random matrices with real entries. With the ensemble made up of standard Gaussian random matrices, i.e. in the circumstance that each element is independently chosen as a real standard Gaussian, Edelman [10] was the first person to obtain results on this problem. The approach taken centred on knowledge of the explicit functional form of the probability density function (PDF) for the event that there are $k$ real eigenvalues denoted $\{\lambda_l\}_{l=1}^k$, and $N-k$ complex eigenvalues denoted $\{x_j \pm iy_j\}_{j=1}^{(N-k)/2}$ with $(x_j, y_j) \in \mathbb{R} \times \mathbb{R}_+$ (the fact that the complex eigenvalues occur in complex conjugate pairs implies $k$ must have the same parity as $N$). Thus it was shown that this is equal to

$$\frac{1}{k!((N-k)/2)!} \frac{1}{Z_N} \left| \Delta\Big( \{\lambda_l\}_{l=1}^k \cup \{x_j \pm iy_j\}_{j=1}^{(N-k)/2} \Big) \right| \prod_{j=1}^k e^{-\lambda_j^2/2}$$
$$\times \prod_{j=1}^{(N-k)/2} 2 e^{y_j^2 - x_j^2} \operatorname{erfc}(\sqrt{2} y_j), \tag{1.1}$$

where $\Delta(\{z_p\}_{p=1}^m) := \prod_{j<l}^m (z_l - z_j)$ denotes the Vandermonde determinant and

$$Z_N = 2^{N(N+1)/4} \prod_{l=1}^N \Gamma(l/2) \tag{1.2}$$

(see also [34]). Integrating (1.1) over $\{\lambda_l\} \cup \{x_j + iy_j\}$ gives the probability $p_{N,k}$ that there are exactly $k$ real eigenvalues. The simplest case to compute is when $k = N$ and thus all eigenvalues are real, for which the probability was found to equal $2^{-N(N-1)/4}$.

Questions relating to the probability that all eigenvalues are real for random matrices with real entries occur in applications. Consider first the tensor structure $\mathcal{A} = (a_{ijk}) \in \mathbb{R}^{p\times p\times 2}$, represented as the column vector $\operatorname{vec}\mathcal{A} \in \mathbb{R}^{4p^2}$. As reviewed in [30], it is of interest to find matrices $U = [\vec{u}_1 \cdots \vec{u}_R] \in \mathbb{R}^{p\times R}$, $V = [\vec{v}_1 \cdots \vec{v}_R] \in \mathbb{R}^{p\times R}$, $W = [\vec{w}_1 \cdots \vec{w}_R] \in \mathbb{R}^{2\times R}$ such that

$$\operatorname{vec}\mathcal{A} = \sum_{r=1}^R \vec{w}_r \otimes \vec{v}_r \otimes \vec{u}_r$$

for $R$ — referred to as the rank — as small as possible. It turns out that with both $(a_{ij1}) =: X_1 \in \mathbb{R}^{p\times p}$ and $(a_{ij2}) =: X_2 \in \mathbb{R}^{p\times p}$ random matrices, entries chosen from a continuous distribution, one has that $R = p$ if all the eigenvalues of $X_1^{-1}X_2$ are real, and $R = p+1$ otherwise [43]. In the Gaussian case these probabilities have been computed in [19] and [7] as equal to $(\Gamma((p+1)/2))^p / G(p+1)$, where $G(x)$ denotes the Barnes $G$-function, and the corresponding large $R$ asymptotic form has been computed in [7].