Linear Algebra and its Applications 510 (2016) 346–360

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Leonard pairs and quantum algebra $U_q(sl_2)$

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ARTICLE INFO

Article history:
Received 28 January 2016
Accepted 29 August 2016
Available online 1 September 2016
Submitted by R. Brualdi

MSC:
17B37
05E30
33C45
33D45

Keywords:
Leonard pair
Quantum algebra
Equitable generator
LB-TD pair

ABSTRACT

Let $\mathbb{K}$ denote an algebraically closed field of characteristic zero. Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. A *Leonard pair* on V is an ordered pair of linear transformations in End(V) such that for each of these transformations there exists a basis for V with respect to which the matrix representing that transformation is diagonal and the matrix representing the other transformation is irreducible tridiagonal. Fix a nonzero scalar $q \in \mathbb{K}$ which is not a root of unity. Consider the quantum algebra $U_q(sl_2)$ with equitable generators $x^{\pm 1}$, y, z. Let d denote a nonnegative integer and let $V_{d,1}$ denote an irreducible $U_q(sl_2)$-module of dimension $d+1$ and of type 1. In this paper, we determine all linear transformations A in End$(V_{d,1})$ such that on $V_{d,1}$, the pair A, x^{-1}, the pair A, y and the pair A, z are all Leonard pairs.

1. Introduction

Leonard pairs were introduced by Terwilliger [10] to extend the algebraic approach of Bannai and Ito [4] to a result of D. Leonard concerning the sequences of orthogonal polynomials with discrete support for which there is a dual sequence of orthogonal

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http://dx.doi.org/10.1016/j.laa.2016.08.034

polynomials. Because these polynomials frequently arise in connection with the finite-dimensional representations of good Lie algebras and quantum groups, it is natural to find Leonard pairs associated with these algebraic objects. Leonard pairs of Krawtchouk type have been described in [8,11] using split basis and normalized semisimple generators of sl_2. Leonard pairs of q-Krawtchouk type have been described in [11] using split basis of $U_q(sl_2)$. Recently, Alnajjar and Curtin [1] gave general construction of Leonard pairs of Racah, Hahn, dual Hahn and Krawtchouk type using equitable basis of sl_2. Alnajjar [2,3] gave general construction of Leonard pairs of q-Racah, q-Hahn, dual q-Hahn, q-Krawtchouk, dual q-Krawtchouk, quantum q-Krawtchouk, and affine q-Krawtchouk type using equitable generators of $U_q(sl_2)$. Equitable presentations for sl_2 and $U_q(sl_2)$ were introduced in [5] and [7], respectively.

In this paper we describe a relationship between Leonard pairs and quantum algebra $U_q(sl_2)$ that appears to be new. Let $\mathbb{K}$ denote an algebraically closed field of characteristic zero. Fix a nonzero scalar $q \in \mathbb{K}$ which is not a root of unity. Consider the quantum algebra $U_q(sl_2)$ with equitable generators $x^{\pm 1}$, y, z. Let d denote a nonnegative integer and let $V_{d,1}$ denote an irreducible $U_q(sl_2)$-module of dimension $d+1$ and of type 1. We determine all linear transformations A in $\mathrm{End}(V_{d,1})$ such that on $V_{d,1}$, the pair A, x^{-1}, the pair A, y and the pair A, z are all Leonard pairs.

2. Preliminaries

In this section we recall the definitions and some related facts concerning Leonard pairs and the quantum algebra $U_q(sl_2)$.

Throughout this paper $\mathbb{K}$ will denote an algebraically closed field of characteristic zero.

2.1. Leonard pairs

In this subsection we recall some terms of Leonard pairs.

Let d be a nonnegative integer. Let $\mathbb{K}^{d+1}$ denote the $\mathbb{K}$-vector space consisting of the column vectors of length $d+1$, and let $\mathrm{Mat}_{d+1}(\mathbb{K})$ denote the $\mathbb{K}$-algebra consisting of the $(d+1) \times (d+1)$ matrices. The algebra $\mathrm{Mat}_{d+1}(\mathbb{K})$ acts on $\mathbb{K}^{d+1}$ by left multiplication.

Let V denote a $\mathbb{K}$-vector space of dimension $d+1$. Let $\mathrm{End}(V)$ denote the $\mathbb{K}$-algebra consisting of all linear transformations from V to V. Let $\{v_i\}_{i=0}^d$ denote a basis for V. For $A \in \mathrm{End}(V)$ and $X \in \mathrm{Mat}_{d+1}(\mathbb{K})$, we say X *represents* A *with respect to* $\{v_i\}_{i=0}^d$ whenever $Av_j = \sum_{i=0}^d X_{ij} v_i$ for $0 \leq j \leq d$.

Let X be a square matrix. X is said to be *upper* (resp. *lower*) *bidiagonal* whenever every nonzero entry appears on or immediately above (resp. below) the main diagonal. X is said to be *tridiagonal* whenever every nonzero entry appears on, immediately above, or immediately below the main diagonal. Assume X is tridiagonal. Then X is said to be *irreducible* whenever all entries immediately above and below the main diagonal are nonzero.