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On the inertia set of a signed tree with loops

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ABSTRACT

A signed graph is a pair (G, Σ), where $G = (V, E)$ is a graph (in which parallel edges and loops are permitted) with $V = \{1, \dots, n\}$ and $\Sigma \subseteq E$. The edges in Σ are called odd edges and the other edges of E even. By $S(G, \Sigma)$ we denote the set of all $n \times n$ real symmetric matrices $A = [a_{i,j}]$ such that if $a_{i,j} < 0$, then among the edges connecting i and j, there must be at least one even edge; if $a_{i,j} > 0$, then among the edges connecting i and j, there must be at least one odd edge; and if $a_{i,j} = 0$, then either there must be at least one odd edge and at least one even edge connecting i and j, or there are no edges connecting i and j. (Here we allow $i = j$.) For a real symmetric matrix A, the partial inertia of A is the pair (p, q), where p and q are the number of positive and negative eigenvalues of A, respectively. If (G, Σ) is a signed graph, we define the inertia set of (G, Σ) as the set of the partial inertias of all matrices $A \in S(G, \Sigma)$. By $\mathrm{MR}(G, \Sigma)$ we denote $\max\{\mathrm{rank}(A) \mid A \in S(G, \Sigma)\}$. We say that a signed graph (G, Σ) satisfies the Northeast Property if for each (p, q) with $p + q < \mathrm{MR}(G, \Sigma)$ in the inertia set of (G, Σ), also $(p + 1, q), (p, q + 1)$ belong to the inertia set of (G, Σ).

In this paper, we show that if (G, Σ) is a signed graph, where G is a tree with possibly loops attached at some of the vertices, then (G, Σ) satisfies the Northeast Property. Furthermore, we present a formula for calculating the inertia set of a signed

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graph (G, Σ), where G is a tree with possibly loops attached at some of the vertices.

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1. Introduction

A signed graph is a pair (G, Σ), where $G = (V, E)$ is a graph (in which parallel edges and loops are permitted) with $V = \{1, \ldots, n\}$ and $\Sigma \subseteq E$. (We refer to [4] for the notions and concepts in graph theory.) The edges in Σ are called odd edges and the other edges of E even edges. By $S(G, \Sigma)$ we denote the set of all $n \times n$ real symmetric matrices $A = [a_{i,j}]$ such that

- if $a_{i,j} < 0$, then among the edges connecting i and j, there must be at least one even edge,
- if $a_{i,j} > 0$, then among the edges connecting i and j, there must be at least one odd edge, and
- if $a_{i,j} = 0$, then either there must be at least one odd edge and at least one even edge connecting i and j, or there are no edges connecting i and j.

Here we allow $i = j$, in which case loops might occur at vertex i. For example the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -2 \\ 0 & -2 & -3 \end{bmatrix}$$

belongs to $S(G, \Sigma)$, where (G, Σ) is the signed graph shown in Fig. 1. For a real symmetric matrix A, the partial inertia of A, denoted by $\operatorname{pin}(A)$, is the pair (p, q), where p and q are the number of positive and negative eigenvalues of A, respectively. If (G, Σ) is a signed graph, we define the *inertia set* of (G, Σ) as the set $\{\operatorname{pin}(A) \mid A \in S(G, \Sigma)\}$; we denote the inertia set of (G, Σ) by $\mathcal{I}(G, \Sigma)$. The *maximum rank* of a signed graph (G, Σ) is $\max\{\operatorname{rank}(A) \mid A \in S(G, \Sigma)\}$ and is denoted by $\operatorname{MR}(G, \Sigma)$.

A *separation* of a graph $G = (V, E)$ is a pair (G_1, G_2) of subgraphs of G such that $G_1 \cup G_2 = G$ and $E(G_1) \cap E(G_2) = \emptyset$; its *order* is the cardinality of $V(G_1) \cap V(G_2)$. If the order of a separation is k, we also say that (G_1, G_2) is a k-separation. The notions of separations transfer without change to signed graphs.

Let $\mathcal{S}, \mathcal{R} \subseteq \mathbb{N}^2$; in this paper we include 0 in the set $\mathbb{N}$. If for each $(p, q) \in \mathcal{S}$, there exists an $(r, s) \in \mathcal{R}$ such that $r \leq p$ and $s \leq q$, then we write $\mathcal{S} \leq \mathcal{R}$. Call a pair

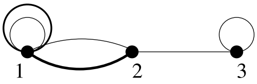

Fig. 1. Thick edges denote odd edges, while thin edges denote even edges.