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A characterization of strongly regular graphs in terms of the largest signless Laplacian eigenvalues

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ABSTRACT

Let G be a simple graph of order n with maximum degree Δ. Let λ (resp. μ) denote the maximum number of common neighbors of a pair of adjacent vertices (resp. nonadjacent distinct vertices) of G. Let $q(G)$ denote the largest eigenvalue of the signless Laplacian matrix of G. We show that

$$q(G) \leq \Delta - \frac{\mu}{4} + \sqrt{\left(\Delta - \frac{\mu}{4}\right)^2 + (1+\lambda)\Delta + \mu(n-1) - \Delta^2},$$

with equality if and only if G is a strongly regular graph with parameters $(n, \Delta, \lambda, \mu)$.

1. Introduction

Let G be a simple graph of order n with maximum degree Δ. Let λ (resp. μ) denote the maximum number of common neighbors of a pair of adjacent vertices (resp. nonadjacent distinct vertices) of G. Then G is called *strongly regular* with parameters $(n, \Delta, \lambda, \mu)$

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if G has order n, every vertex of G has the same degree Δ, and every chosen pair of adjacent vertices (resp. nonadjacent distinct vertices) has the same number λ (resp. μ) of common neighbors. Strongly regular graphs receive attentions from many different areas of researchers. For example, strongly regular graphs form the simplest nontrivial class of distance-regular graphs [2], or more generally of association schemes [1]. Any regular connected graph with three distinct eigenvalues must be strongly regular graphs [3, Theorem 9.1.2]. See [12, Chapter 21] for many beautiful properties of strongly regular graphs. This note aims to characterize strongly regular graphs in terms of the graph parameters n, Δ, λ, μ and an additional parameter $q(G)$, the largest eigenvalue of the signless Laplacian matrix associated with G. Theorem 3.1 is our main result.

2. Preliminaries

In this section we shall introduce notations and basic properties that are needed in the statement and proof of our main result. For more reading, the reader may refer to [3].

Throughout this note, let $G = (V, E)$ be a simple graph of order n with vertex set $V = \{1, 2, \ldots, n\}$ and edge set E. The *adjacency matrix* $A(G)$ of G is an n-by-n 01-matrix whose ij-entry is 1 or 0 according to whether i and j are adjacent or not adjacent respectively. Then the parameters λ and μ defined in the previous section can be restated as

$$\lambda = \max_{j\neq k, jk\in E}(A(G)^2)_{jk}, \qquad \mu = \max_{j\neq k, jk\notin E}(A(G)^2)_{jk}.$$

The *degree* d_i of a vertex $i \in V$ is the number $|N(i)|$, where $N(i)$ is the set of neighbors of i. The number $\Delta = \max_{1\leq i\leq n} d_i$ is called the *maximum degree* of G. Note that $\mu = 0$ implies that $\Delta = \lambda + 1$, and G is a union of cliques, among them the maximal one has order $\Delta + 1$. Let $\rho(A(G))$ denote the largest eigenvalue of $A(G)$. Then the Rayleigh principle implies that $x^T A(G)x \leq \rho(A(G))x^T x$, where x is a column vector of length n. By the Perron–Frobenius Theorem, $\rho(A(G)) \leq \Delta$ [13, Chapter 2]. Let $D(G) = \operatorname{diag}(d_1, d_2, \ldots, d_n)$ be the diagonal matrix with entries $d_1, d_2, \ldots, d_n$ in the diagonal. Then the matrices $Q(G) = D(G)+A(G)$ and $L(G) = D(G)-A(G)$ are called the *signless Laplacian matrix* and *Laplacian matrix* of G respectively. It was checked in small graphs by D. Cvetković and S.K. Simić, and they found that less graphs have the same set of signless Laplacian eigenvalues than have that of adjacency eigenvalues and that of Laplacian eigenvalues. With this strong basis, it is believed that studying graphs using signless Laplacian eigenvalues is more efficient than studying them by other eigenvalues associated with graphs [5]. In this note we focus on signless Laplacian matrices, so we call the eigenvalues of $Q(G)$ the *eigenvalues* of G.

Obviously $Q(G)$ is symmetric, so has n real eigenvalues. Let $q(G)$ denote the largest eigenvalue of $Q(G)$. Note that $q(G) \geq \Delta + 1$ [9]. As given in [3, Corollary 3.9.3], a short proof of the previous lower bound of $q(G)$ is by using interlacing property to show that $q(G) \geq q(K_{1,\Delta}) = \Delta + 1$, where $K_{1,\Delta}$ is a subgraph of G of order $\Delta + 1$ and