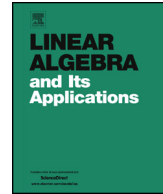




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## Compact sets in the free topology

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## ABSTRACT

Subsets of the set of  $g$ -tuples of matrices that are closed with respect to direct sums and compact in the free topology are characterized. They are, in a dilation theoretic sense, the hull of a single point.

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## 1. Introduction

Given positive integers  $n, g$ , let  $M_n(\mathbb{C})^g$  denote the set of  $g$ -tuples of  $n \times n$  matrices. Let  $M(\mathbb{C})^g$  denote the sequence  $(M_n(\mathbb{C})^g)_n$ . A subset  $E$  of  $M(\mathbb{C})^g$  is a sequence  $(E(n))$  where  $E(n) \subset M_n(\mathbb{C})^g$ . The free topology [1] has as a basis sets of the form  $G_\delta = (G_\delta(n)) \subset M(\mathbb{C})^g$ , where

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$$G_\delta(n) = \{X \in M_n(\mathbb{C})^g : \|\delta(X)\| < 1\},$$

and  $\delta$  is a (matrix-valued) free polynomial. Agler and McCarthy [1] prove the remarkable result that a bounded free function on a basis set  $G_\delta$  is uniformly approximable by polynomials on each smaller set of the form

$$G_\delta \supset K_\delta^s = \{X \in M(\mathbb{C})^g : \|\delta(X)\| \leq s\}, \quad 0 \leq s < 1.$$

See [3] for a generalization of this result. For the definitive treatment of free function theory, see [6].

Sets  $E \subset M(\mathbb{C})^g$  naturally arising in free analysis ([2,4,5,7–9,12] is a sampling of the references not already cited) are typically closed with respect to direct sums in the sense that if  $X \in E(n)$  and  $Y \in E(m)$ , then

$$X \oplus Y = \left( \left( \begin{matrix} X_1 & 0 \\ 0 & Y_1 \end{matrix} \right), \dots, \left( \begin{matrix} X_g & 0 \\ 0 & Y_g \end{matrix} \right) \right) \in E(n+m).$$

**Theorem 1.1** below, characterizing free topology compact sets that are closed with respect to direct sums, is the main result of this article. A tuple  $Y \in M_n(\mathbb{C})^g$  **polynomially dilates** to a tuple  $X \in M_N(\mathbb{C})^g$  if there is an isometry  $V : \mathbb{C}^n \rightarrow \mathbb{C}^N$  such that for all free polynomials  $p$ ,

$$p(Y) = V^*p(X)V.$$

An **ampliation** of  $X$  is a tuple of the form  $I_k \otimes X$ , for some positive integer  $k$ . The **polynomial dilation hull** of  $X \in M(\mathbb{C})^g$  is the set of all  $Y$  that dilate to an ampliation of  $X$ .

**Theorem 1.1.** *A nonempty subset  $E$  of  $M(\mathbb{C})^g$  that is closed with respect to direct sums is compact if and only if it is the polynomial dilation hull of an  $X \in E$ .*

**Corollary 1.2.** *If  $E \subset M(\mathbb{C})^g$  is closed with respect to direct sums and is compact in the free topology, then there exists a free polynomial  $p$  such that  $E$  is a subset of the zero set of  $p$ ; i.e.,  $p(Y) = 0$  for all  $Y \in E$ . In particular, there is an  $N$  such that for  $n \geq N$  the set  $E(n)$  has empty interior.*

**Proof.** By **Theorem 1.1**, there is an  $n$  and  $X \in E(n)$  such that each  $Y \in E$  polynomially dilates to an ampliation of  $X$ . Choose a nonzero (scalar) free polynomial  $p$  such that  $p(X) = 0$  (using the fact that the span of  $\{w(X) : w \text{ is a word}\}$  is a subset of the finite dimensional vector space  $M_n(\mathbb{C})$ ). It follows that  $p(Y) = 0$  for all  $Y$ . Hence  $E$  is a subset of the zero set of  $p$ . It is well known (see for instance the Amistur–Levitzki Theorem [11]) that the zero set  $p$  in  $M_n(\mathbb{C})^g$  must have empty interior for sufficiently large  $n$ .  $\square$

The authors thank Igor Klep for a fruitful correspondence that led to this article. The proof of **Theorem 1.1** occupies the remainder of this article.

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