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Compact sets in the free topology

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ABSTRACT

Subsets of the set of g-tuples of matrices that are closed with respect to direct sums and compact in the free topology are characterized. They are, in a dilation theoretic sense, the hull of a single point.

1. Introduction

Given positive integers n, g, let $M_n(\mathbb{C})^g$ denote the set of g-tuples of $n \times n$ matrices. Let $M(\mathbb{C})^g$ denote the sequence $(M_n(\mathbb{C})^g)_n$. A subset E of $M(\mathbb{C})^g$ is a sequence $(E(n))$ where $E(n) \subset M_n(\mathbb{C})^g$. The free topology [1] has as a basis sets of the form $G_\delta = (G_\delta(n)) \subset M(\mathbb{C})^g$, where

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$$G_\delta(n) = \{X \in M_n(\mathbb{C})^g : \|\delta(X)\| < 1\},$$

and δ is a (matrix-valued) free polynomial. Agler and McCarthy [1] prove the remarkable result that a bounded free function on a basis set G_δ is uniformly approximable by polynomials on each smaller set of the form

$$G_\delta \supset K_\delta^s = \{X \in M(\mathbb{C})^g : \|\delta(X)\| \le s\}, \quad 0 \le s < 1.$$

See [3] for a generalization of this result. For the definitive treatment of free function theory, see [6].

Sets $E \subset M(\mathbb{C})^g$ naturally arising in free analysis ([2,4,5,7–9,12] is a sampling of the references not already cited) are typically closed with respect to direct sums in the sense that if $X \in E(n)$ and $Y \in E(m)$, then

$$X \oplus Y = \left(\begin{pmatrix} X_1 & 0 \\ 0 & Y_1 \end{pmatrix}, \dots, \begin{pmatrix} X_g & 0 \\ 0 & Y_g \end{pmatrix} \right) \in E(n+m).$$

Theorem 1.1 below, characterizing free topology compact sets that are closed with respect to directs sums, is the main result of this article. A tuple $Y \in M_n(\mathbb{C})^g$ **polynomially dilates** to a tuple $X \in M_N(\mathbb{C})^g$ if there is an isometry $V : \mathbb{C}^n \to \mathbb{C}^N$ such that for all free polynomials p,

$$p(Y) = V^* p(X) V.$$

An **ampliation** of X is a tuple of the form $I_k \otimes X$, for some positive integer k. The **polynomial dilation hull** of $X \in M(\mathbb{C})^g$ is the set of all Y that dilate to an ampliation of X.

Theorem 1.1. *A nonempty subset E of $M(\mathbb{C})^g$ that is closed with respect to direct sums is compact if and only if it is the polynomial dilation hull of an $X \in E$.*

Corollary 1.2. *If $E \subset M(\mathbb{C})^g$ is closed with respect to direct sums and is compact in the free topology, then there exists a free polynomial p such that E is a subset of the zero set of p; i.e., $p(Y) = 0$ for all $Y \in E$. In particular, there is an N such that for $n \ge N$ the set $E(n)$ has empty interior.*

Proof. By Theorem 1.1, there is an n and $X \in E(n)$ such that each $Y \in E$ polynomially dilates to an ampliation of X. Choose a nonzero (scalar) free polynomial p such that $p(X) = 0$ (using the fact that the span of $\{w(X) : w \text{ is a word}\}$ is a subset of the finite dimensional vector space $M_n(\mathbb{C})$). It follows that $p(Y) = 0$ for all Y. Hence E is a subset of the zero set of p. It is well known (see for instance the Amistur–Levitzki Theorem [11]) that the zero set p in $M_n(\mathbb{C})^g$ must have empty interior for sufficiently large n. □

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