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Beyond graph energy: Norms of graphs and matrices

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A B S T R A C T

In 1978 Gutman introduced the energy of a graph as the sum of the absolute values of graph eigenvalues, and ever since then graph energy has been intensively studied.
Since graph energy is the trace norm of the adjacency matrix, matrix norms provide a natural background for its study. Thus, this paper surveys research on matrix norms that aims to expand and advance the study of graph energy.
The focus is exclusively on the Ky Fan and the Schatten norms, both generalizing and enriching the trace norm. As it turns out, the study of extremal properties of these norms leads to numerous analytic problems with deep roots in combinatorics.
The survey brings to the fore the exceptional role of Hadamard matrices, conference matrices, and conference graphs in matrix norms. In addition, a vast new matrix class is studied, a relaxation of symmetric Hadamard matrices.
The survey presents solutions to just a fraction of a larger body of similar problems bonding analysis to combinatorics. Thus, open problems and questions are raised to outline topics for further investigation.

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1. Introduction

This paper overviews the current research on the Ky Fan and the Schatten matrix norms that aims to expand and push forward the study of graph energy.

Graph energy has been introduced by Gutman in 1978 [13] as the sum of the absolute values of the graph eigenvalues; since then its study has produced a monumental body of work, as witnessed by the references of the monograph [14]. One reckons that such an enduring interest must be caused by some truly special property of the graph energy parameter.

1.1. Graph energy as a matrix norm

To crack the mystery of graph energy, it may be helpful to view it as the trace norm of the adjacency matrix. Recall that the *trace norm* $\|A\|_*$ of a matrix A is the sum of its singular values, which for a real symmetric matrix are just the moduli of its eigenvalues.

Hence, if G is graph with adjacency matrix A, then the energy of G is the trace norm of A.

This simple observation, made in [31], triggered some sort of a chain reaction. On the one hand, usual tools for norms provided new techniques for graph energy; see, e.g., [2,9,10,42]. On the other hand, viewed as a trace norm, graph energy was extended to non-symmetric and even to non-square matrices, and gave rise to new topics like "skew energy" and "incidence energy", see, e.g., [1,15–17,45,46,48].

To follow this track further, we need a few definitions. Write $M_{m,n}$ for the space of the $m \times n$ complex matrices.

Definition 1.1. A **matrix norm** is a nonnegative function $\|\cdot\|$ defined on $M_{m,n}$ such that:
(a) $\|A\| = 0$ if and only if $A = 0$;
(b) $\|cA\| = |c| \|A\|$ for every complex number c;
(c) $\|A + B\| \leq \|A\| + \|B\|$ for every $A, B \in M_{m,n}$.

Observe that the usual definition of matrix norm includes additional properties, but they are not used in this survey.

A simple example of a matrix norm is the *max-norm* $\|A\|_{\max}$ defined for any matrix $A := [a_{i,j}]$ as $\|A\|_{\max} := \max_{i,j} |a_{i,j}|$.

Another example of a matrix norm is the largest singular value of a matrix, also known as its *operator norm*.

By contrast, the spectral radius $\rho(A)$ of a square matrix A is not a matrix norm. Likewise, if $\lambda_1(A), \ldots, \lambda_n(A)$ are the eigenvalues of A, neither of the functions

$$g(A) := |\lambda_1(A)| + \cdots + |\lambda_n(A)| \quad \text{and} \quad h(A) := |\text{Re}\,\lambda_1(A)| + \cdots + |\text{Re}\,\lambda_n(A)|$$

is a matrix norm. Indeed, if A is an upper triangular matrix with zero diagonal, then $\rho(A) = g(A) = h(A) = 0$, violating condition (a). Worse yet, if A is the adjacency matrix