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Proof of a conjecture on 'plateaux' phenomenon of graph Laplacian eigenvalues

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ABSTRACT

Let G be a simple graph. A pendant path of G is a path such that one of its end vertices has degree 1, the other end has degree ≥ 3, and all the internal vertices have degree 2. Let $p_k(G)$ be the number of pendant paths of length k of G, and $q_k(G)$ be the number of vertices with degree ≥ 3 which are an end vertex of some pendant paths of length k. Motivated by the problem of characterizing dendritic trees, N. Saito and E. Woei conjectured that any graph G has some Laplacian eigenvalue with multiplicity at least $p_k(G)-q_k(G)$. We prove a more general result for both Laplacian and signless Laplacian eigenvalues from which the conjecture follows.

1. Introduction

Let G be a simple graph. A *pendant path* of G is a path such that one of its end vertices has degree 1, the other end has degree ≥ 3, and all the internal ver-

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tices have degree 2. Let $p_k(G)$ denote the number of pendant paths of length k of G, and $q_k(G)$ denote the number of vertices with degree ≥ 3 which are an end vertex of some pendant paths of length k. Saito and Woei [4] studied Laplacian eigenvalues of dendritic trees. They observed eigenvalue(s) 'plateaux' (i.e., set of eigenvalue(s) with multiplicity) in the Laplacian eigenvalues of such trees. More generally, they showed that $(3 \pm \sqrt{5})/2$ is a Laplacian eigenvalue of any graph G with multiplicity at least $p_2(G) - q_2(G)$. This motivated them to put forward the following conjecture.

Conjecture 1. *([4]) For any positive integer k, any graph G has some Laplacian eigenvalue with multiplicity at least $p_k(G) - q_k(G)$.*

We remark that the special cases $k = 1$ follows from a result of [2] (see also [3]) asserting that multiplicity of 1 as a Laplacian eigenvalue of a graph G is at least $p_1(G) - q_1(G)$. We prove a more general result (Theorem 4 below) for both Laplacian and signless Laplacian eigenvalues from which Conjecture 1 follows.

2. Preliminaries

Let G be a simple graph with vertex set $\{v_1, \ldots, v_n\}$ and edge set $\{e_1, \ldots, e_m\}$. The *adjacency matrix* of G is an $n \times n$ matrix $A = A(G)$ whose (i, j)-entry is 1 if v_i is adjacent to v_j and 0 otherwise. The *incident matrix* of G, $X = X(G) = (x_{ij})$, is an $n \times m$ matrix whose rows and columns are indexed by vertex set and edge set of G, respectively, where

$$x_{ij} = \begin{cases} 1 & \text{if } e_j \text{ is incident with } v_i, \\ 0 & \text{otherwise.} \end{cases}$$

If we orient the edges of G, we may define similarly the *directed incidence matrix* $D = D(G) = (d_{ij})$, with respect to the particular orientation, as

$$d_{ij} = \begin{cases} +1 & \text{if } e_j \text{ is an incoming edge to } v_i, \\ -1 & \text{if } e_j \text{ is an outgoing edge from } v_i, \\ 0 & \text{otherwise.} \end{cases}$$

The matrices $L(G) := DD^\top$ and $Q(G) := XX^\top$ are called the *Laplacian matrix* and *signless Laplacian matrix* of G, respectively. It is easily seen that $L(G) = \Delta - A$ and $Q(G) = \Delta + A$ where Δ is the diagonal matrix of vertex degrees of G.

For any $n \times m$ matrix M, and nonzero real λ, we have

$$\det \begin{pmatrix} \lambda I_n & M \\ M^\top & \lambda I_m \end{pmatrix} = \lambda^{m-n} \det(\lambda^2 I_n - MM^\top).$$